Constructive Ordinal Exponentiation in Homotopy Type Theory

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Constructive ordinals Ordinals are a powerful tool for establishing consistency of logical theories, proving termination of processes and justifying induction and recursion. Constructively, there are many different approaches to ordinals, such as ordinal notation systems [6], or Brouwer trees [4], or as wellfounded trees with finite or countable branchings [5, 1]. The homotopy type theory book follows the classical idea of considering ordinals as order types of well ordered sets, and defines ordinals as types equipped with an order relation that is transitive, extensional (elements with the same predecessors are equal) and wellfounded (the order admits the principle of transfinite induction) [9, §10.3]. Notably, the univalence axiom is used to exhibit the type of (small) ordinals as a (large) ordinal; specifically, it is used to show that the relation on ordinals given by bounded simulations is extensional. This gives rise to a fascinating theory of ordinals, often skirting the edges of what is constructively achievable. With this in mind, is it possible to develop a constructive theory of arithmetic for such ordinals, with operations of addition, multiplication and exponentiation which extend the usual arithmetic for the natural numbers?

Ordinal exponentiation via case distinction Addition and multiplication can be realised by disjoint union and Cartesian product of the underlying types of the ordinals, respectively. Their basic properties were investigated by Escardó [3].

The case of exponentiation is constructively more challenging. From the classical theory of ordinals, we know what the specification should be: for zero and successors, exponentiation should be repeated multiplication, and it should be continuous as soon as the base is non-zero:

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta \times \alpha \\ \alpha^{\sup_{i:I} f(i)} &= \sup_{i:I} \alpha^{f(i)} \quad \text{(for } \alpha \neq 0,\ I \text{ inhabited)} \\ 0^\beta &= 0 \qquad\qquad\qquad\quad \text{(for } \beta \neq 0) \end{aligned} \tag{†}$$

Using classical logic, this is already a definition of exponentiation, but not so in a constructive setting, where the ability to make definitions by case distinctions on arbitrary ordinals is not available. In fact, we can show:

Theorem 1. *There is an operation α^β satisfying the specification (†) for all ordinals α and β if and only if excluded middle holds.*

In fact, excluded middle follows as soon as there is an exponentiation operator which is monotone in the exponent, and satisfies the first two equations of the specification (†). There is thus no hope of defining ordinal exponentiation constructively for arbitrary ordinals.

Ordinal exponentiation as functions with finite support However, we could still hope to define exponentiation for restricted classes of ordinals. For $\alpha > 0$ (which is equivalent to α having a least element $\bot$), Sierpiński [8, §XIV.15] gives an explicit construction of the exponential α^β as the collection of functions $\beta \to \alpha$ with *finite support*, i.e., functions $f : \beta \to \alpha$ such that $f(x) > \bot$ for only finitely many x. While this definition works well classically, the order relation it induces does not seem to be well-behaved constructively. The usual classical argument that the exponential is an ordinal requires decidability of the order on α, and decidable equality on α seems to be required to verify the expected properties (such as the specification (†)) of this ordinal. In general, neither of these assumptions are constructively justified.

Constructive exponentiation for ordinals with a detachable least element Let α be an ordinal of the form $\alpha = 1 + \gamma$ for some ordinal γ. That is, let α be an ordinal with a least element which is detachable — we can decide if a given element is the least one or not. For such α, we are able to define the exponential α^β constructively, by considering a "combinatorial" variant of Sierpiński's construction.

Since β is an ordinal, we can think of a finitely supported function from β to γ as a finite list of output-input[1] pairs $[(c_0, b_0), (c_1, b_1), \ldots, (c_n, b_n)] : \mathsf{List}\,(\gamma \times \beta)$ which is strictly decreasing in the second argument (to enforce that each input has a unique output), with all inputs not occurring in the list being sent to the least element. Write

$$\mathsf{D_2List}(\gamma, \beta) :\equiv \big(\Sigma\, \ell : \mathsf{List}\,(\gamma \times \beta)\big)\ \mathsf{is\text{-}decreasing}\,(\mathsf{map}\,\pi_2\,\ell)$$

for the type of such lists of pairs decreasing in the second component. The idea of the combinatorial presentation using lists is similar to—but more general than—Setzer's sketch [7, App. A] of the construction of exponentials with base ω.

Theorem 2. *The type $\mathsf{D_2List}(\gamma, \beta)$ is an ordinal, when ordered lexicographically. Moreover, it satisfies the specification (†) for $\alpha = 1 + \gamma$.*

Proof sketch. Because the list is decreasing, the lexicographic order is wellfounded.

For verifying the specification (†), note that the only list with elements from $\gamma \times \mathbf{0}$ is the empty list, so $\mathsf{D_2List}(\gamma, \mathbf{0}) = \mathbf{1}$. For checking that $\mathsf{D_2List}(\gamma, \beta + \mathbf{1}) = \mathsf{D_2List}(\gamma, \beta) \times (1 + \gamma)$, note that a list ℓ in $\mathsf{D_2List}(\gamma, \beta + \mathbf{1})$ contains at most one head of the form $(c_0, \mathsf{inr}\,\star)$, followed by a list ℓ_1 in $\mathsf{D_2List}(\gamma, \beta)$. If the head is of the form $(c_0, \mathsf{inr}\,\star)$, the list ℓ corresponds to the pair $(\ell_1, \mathsf{inr}\,c_0)$, and otherwise it corresponds to the pair $(\ell, \mathsf{inl}\,\star)$. For the supremum case, it is crucial that the lists are decreasing in the second component; see the Agda code for details. □

Alternative constructions of exponentials In work in progress, we are investigating alternative definitions of exponentials. In particular, building on a suggestion by David Wärn, we consider the following definition by transfinite recursion:

$$\alpha^\beta :\equiv \sup_{\mathbf{1}+\beta} \begin{cases} \mathsf{inl}\,\star \mapsto \mathbf{1} \\ \mathsf{inr}\,b \mapsto \alpha^{\beta \downarrow b} \times \alpha, \end{cases}$$

where $\beta \downarrow b$ is the initial segment of β consisting of elements strictly smaller than b. This is motivated by the specification (†) and the observation that every ordinal γ is the supremum of the successors of its initial segments, i.e., $\gamma = \sup_{c:\gamma}((\gamma \downarrow c) + \mathbf{1})$. This operation indeed satisfies the specification (†) for $\alpha \geq 1$. Relating this construction to the one above is ongoing work.

[1] We prefer to use output-input pairs rather than input-output pairs so that their order corresponds to the usual order on the product $\gamma \times \beta$, which is reverse lexicographic.

Formalisation We have formalised our results in Agda, building on Escardó's TypeTopology development [2]. We have found Agda extremely valuable in developing our proofs as the intensional nature of our construction makes for rather combinatorial arguments. The source code can be found at `https://github.com/fredrikNordvallForsberg/TypeTopology/blob/exponentiation/source/Ordinals/Exponentiation/`.

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