Constructive Notions of Ordinals in Homotopy Type Theory

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TYPES'21, online, 14–18 June 2021

Motivation

Ordinals are fundamental and useful, e.g. for

- proving termination; or
- justifying induction and recursion.

Unfortunately: constructively problematic.

Classical notion fragments into disconnected notions, each with pros and cons.

We consider three constructive notions in HoTT, and relate them to each other.

Extensional Wellfounded Orders

Following the HoTT book and Escardó, and inspired by Taylor:

Definition

The type Ord consists of pairs $(X : \text{Type}, \prec : X \to X \to \text{Prop})$ such that:

- $\prec$ is transitive
  - $x \prec y \to y \prec z \to x \prec z$;
- $\prec$ is extensional
  - elements with the same $\prec$-predecessors are equal;
- $\prec$ is wellfounded
  - every element is accessible, where x is accessible if every $y \prec x$ is accessible.

An Order on Extensional Wellfounded Orders

Let $(X, \prec_X)$, $(Y, \prec_Y)$: Ord.

$X \leq Y$ is the type of monotone functions $f : X \to Y$ satisfying a *simulation condition*: if $y \prec_Y f\, x$, then we have an $x_0 \prec_X x$ such that $f\, x_0 = y$.

$X < Y$ is the type of *bounded* simulations, i.e. those inducing an equivalence

$$X \simeq \text{“initial segment of } Y \text{ below } y\text{”}$$

for some $y : Y$.

Brouwer Trees

Consider the usual inductive type $\mathcal{O}$ of Brouwer trees:

$$\mathsf{zero} : \mathcal{O} \qquad \mathsf{succ} : \mathcal{O} \to \mathcal{O} \qquad \mathsf{sup} : (\mathbb{N} \to \mathcal{O}) \to \mathcal{O}$$

Problem: we do *not* have $\mathsf{sup}(s_0 s_1 s_2 \ldots) = \mathsf{sup}(s_1 s_0 s_2 \ldots)$.

Our notion: a type of Brouwer trees that can

(i) faithfully represent ordinals, and

(ii) classify an ordinal as *zero*, *successor* or a *limit*,

Brouwer Trees as a Quotient Inductive-Inductive Type

Definition

We mutually construct a type Brw : Set and a relation $\leq$: Brw $\to$ Brw $\to$ Prop:

- The constructors of Brw include

  $$\begin{aligned}
  &\mathsf{zero} : \mathsf{Brw} \\
  &\mathsf{succ} : \mathsf{Brw} \to \mathsf{Brw} \\
  &\mathsf{limit} : (\mathbb{N} \xrightarrow{<} \mathsf{Brw}) \to \mathsf{Brw} && \text{(for strictly increasing sequences)} \\
  &\mathsf{bisim} : f \approx^{\leq} g \to \mathsf{limit}\, f = \mathsf{limit}\, g && (f \text{ and } g \text{ are bisimilar})
  \end{aligned}$$

  where $x < y$ stands for succ $x \leq y$.

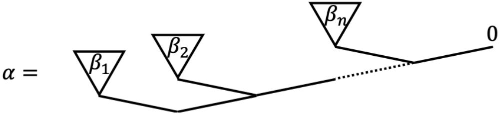
- The constructors for $\leq$ ensure transitivity, that zero is minimal, that succ is monotone, and that limit f is the least upper bound of f.

Cantor Normal Forms as a Subset of Binary Trees

Motivation: $\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_n}$ with $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$

Definition

- Let $\mathcal{T}$ be the type of *unlabeled binary trees*: $0 : \mathcal{T}$, $\omega^{-} + - : \mathcal{T} \to \mathcal{T} \to \mathcal{T}$.



- Let $<$ be the *lexicographical order* on $\mathcal{T}$.
- Define $\mathsf{isCNF}(\alpha)$ to express $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$.

We write $\mathsf{Cnf} :\equiv \Sigma(t : \mathcal{T}).\mathsf{isCNF}(t)$ for the type of *Cantor normal forms*.

Similarities

Ord, Brw, Cnf …

- are wellfounded: all elements accessible
- are extensional: $(\forall z. z < x \leftrightarrow z < y) \rightarrow x = y$
- have addition and multiplication
  - **and these satisfy the same specifications**
    (e.g. are continuous in the second argument)!

Differences and Connections

$$
\begin{tikzcd}[column sep=huge]
\underset{\text{decidable}}{\boxed{\mathsf{Cnf}}} \arrow[r, hook, "\mathsf{CtoB}", "(\omega^a + b) \mapsto \omega^{\mathsf{CtoB}(a)} + \mathsf{CtoB}(b)"'] & \underset{\text{partially decidable}}{\boxed{\mathsf{Brw}}} \arrow[r, hook, "\mathsf{BtoO}", "A \mapsto \Sigma(Y : \mathsf{Brw}).Y < A"'] & \underset{\text{undecidable}}{\boxed{\mathsf{Ord}}}
\end{tikzcd}
$$

- injective
- preserves and reflects $<$, $\leq$
- commutes with $+$, $*$, ω^x
- bounded (by ϵ_0)

- injective
- preserves $<$, $\leq$
- over-approximates $+$, $*$:
  $\mathsf{BtoO}(x+y) \geq \mathsf{BtoO}(x) + \mathsf{BtoO}(y)$
- commutes with limits
  (but not successors)
- BtoO is a simulation $\Rightarrow$ WLPO
- LEM $\Rightarrow$ BtoO is a simulation
- bounded (by Brw)

paper: Connecting Constructive Notions of Ordinals in Homotopy Type Theory

THANK YOU!