On Symmetries of Spheres in HoTT-UF

Pierre Cagne[*] Nicolai Kraus[†] Marc Bezem[*]

The goal of this talk is to give insights in the symmetries of the n-sphere in synthetic homotopy theory, i.e. the type $S^n = S^n$.

We work in intuitionistic Martin-Löf's type theory with Σ-, Π- and Id-types and with a cumulative hierarchy of universes, simply written $\mathbb{U}$, for which Voevodsky's univalence axiom hold. A good reference for our setting is the HoTT Book [Uni13], to which we will refer frequently. We specifically use the following higher inductive types (HITs): the propositional truncation $\|A\|$ of an arbitrary type A ([Uni13, Ch. 3.7]); the set truncation $\|A\|_0$ of an arbitrary type A ([Uni13, Ch. 6.9]); the circle S^1 ([Uni13, Ch. 6.4]); the suspension ΣA of an arbitrary type A ([Uni13, Ch. 6.5]).

We give some more details of the circle and the suspension, as they are crucial for our presentation. The circle is a higher inductive type with one point constructor and one path constructor:

$$\begin{aligned} \bullet &: S^1, \quad \text{the base point} \\ \circlearrowleft &: \bullet =_{S^1} \bullet, \quad \text{the loop} \end{aligned}$$

The circle comes with an elimination rule such that functions $S^1 \to A$ correspond to pairs of an element $a : A$ and a path $\ell : a = a$ for any type A. The first part of the talk will be dedicated to showing that

$$(S^1 = S^1) \simeq (S^1 + S^1). \tag{1}$$

For $n \geq 2$, the n-sphere S^n is inductively defined as the suspension $\Sigma(S^{n-1})$. The suspension ΣA of a type A is a higher inductive type with two point constructors and path constructors indexed by A:

$$\begin{aligned} N &: \Sigma A, \quad \text{the `North pole'} \\ S &: \Sigma A, \quad \text{the `South pole'} \\ \mathit{merid} &: A \to (N =_{\Sigma A} S), \quad \text{the `meridians'} \end{aligned}$$

The elimination principle for ΣA gives again a correspondence between the type $\Sigma A \to B$ and the type of triplets consisting of $b_N : B$, $b_S : B$ and $m : A \to b_N = b_S$.

One cannot expect (1) to generalise to higher dimensions. However, (1) implies that $S^1 = S^1$ consists of two equivalent connected components. Modulo univalence, one component contains id_{S^1} and the other $-\mathrm{id}_{S^1}$. The latter is the function $S^1 \to S^1$ corresponding to the pair $\bullet : S^1$ together with the path $\circlearrowleft^{-1} : \bullet = \bullet$. This weaker statement does in fact generalise to higher

[*]Universitetet i Bergen
[†]University of Birmingham

spheres in the homotopy theory of topological spaces. In the talk, we will elaborate a proof in HoTT-UF for the case $n = 2$, along the following lines.

The first step is to define $-\mathrm{id}_{\Sigma A}$ for any type A as the function corresponding to the triplet $S : \Sigma A$, $N : \Sigma A$ and $\mathit{merid}(\_)^{-1} : (A \to S = N)$. In other words, $-\mathrm{id}_{\Sigma A}$ flips the poles and reverses each meridian. Notice that $-\mathrm{id}_{\Sigma A}$ is an equivalence, as it is its own pseudo-inverse. The function

$$\mathrm{flip}_A :\equiv \_ \circ -\mathrm{id}_{\Sigma A} : (\Sigma A \to \Sigma A) \to (\Sigma A \to \Sigma A)$$

is then an equivalence, hence establishing an equivalence from the connected component at $\mathrm{id}_{\Sigma A}$ to the connected component at $-\mathrm{id}_{\Sigma A}$. Notice that $-\mathrm{id}_{\Sigma A}$ is not necessarily distinct from $\mathrm{id}_{\Sigma A}$ (take for example $A :\equiv 1$ for which ΣA is contractible). In particular, it is non-trivial to prove that $-\mathrm{id}_{S^2} \neq \mathrm{id}_{S^2}$.

As the sphere S^2 is connected, in proving the proposition $\|\varphi = \mathrm{id}_{S^2}\| + \|\varphi = -\mathrm{id}_{S^2}\|$ for an equivalence $\varphi : S^2 \simeq S^2$, one can as well suppose that φ is a pointed map by a path $\varphi_0 : N = \varphi(N)$. It is worth computing the degree of such a φ. Recall that the degree $d(f, f_0)$ of a pointed function $(f, f_0) : S^2 \to_* S^2$ is defined as the integer $\bar{\pi}_2(f, f_0)(1)$ where $\bar{\pi}_2(f, f_0)$ is the group morphism $\pi_2(f, f_0) : \pi_2(S^2) \to \pi_2(S^2)$ viewed through the equivalence $\pi_2(S^2) \simeq \mathbb{Z}$.

For example, for each $k : \mathbb{Z}$, the following map $\delta_k : S^2 \to_* S^2$ has degree k: first define $c_k : S^1 \to S^1$ as the map that corresponds to the pair $\bullet : S^1$ together with the path $\circlearrowleft^k : \bullet = \bullet$; then define δ_k as the map corresponding to the triplet $N : S^2$, $S : S^2$ and $\mathit{merid} \circ c_k : S^1 \to N = S$, obviously pointed by the path $\mathit{refl}_N : N = \delta_k(N)$. It is easy to see that $\delta_1 = \mathrm{id}_{S^2}$ and one can prove that $\delta_{-1} = -\mathrm{id}_{S^2}$ also. Proving that δ_k has indeed degree k is non-trivial, and we shall exhibit a proof using the Hopf fibration.

Using the functoriality of π_2, one gets

$$d((g, g_0) \circ (f, f_0)) = \bar{\pi}_2(g, g_0)\,(\bar{\pi}_2(f, f_0)(1)) = \bar{\pi}_2(g, g_0)(1) \times \bar{\pi}_2(f, f_0)(1).$$

The last identity comes from the fact that $\bar{\pi}_2(g, g_0)$ is a group morphism $\mathbb{Z} \to \mathbb{Z}$. In other words, d is a morphism of monoids, and as such, it maps equivalences to invertible elements of $\mathbb{Z}$. Hence, $d(\varphi, \varphi_0) = \pm 1$. The last step is to prove that having the same degree means precisely being in the same connected component of $S^2 \to_* S^2$. In order to do so, we shall give an alternate description of the degree, based on the Hopf fibration, and on the proof that $\pi_2(S^2) \simeq \mathbb{Z}$ (cf. [Uni13, Ch. 8.6]).

This result generalizes to the case $n > 2$ with the help of the Freudenthal suspension theorem ([Uni13, Ch. 8.6]). If time permits, we will sketch our path to a full proof of the fact that $S^n = S^n$ has exactly two connected components.

Future works include formalizing this proof in cubical type theory (CTT) and experimenting with actual computation of the degree of selected symmetries. This is one of the motivations for this work. Indeed, the univalence axiom is necessary for the definition of the Hopf fibration and for the Freudenthal suspension theorem (and hence for the definition of the degree), and an implementation in CTT will display the computational content of univalence at work. Hopefully, this would be a feasible computational challenge, simpler than the computation of Brunerie's number (cf. [Bru16, Corollary 3.4.5]), which is still out of reach of CTT and other systems.

[Bru16] Guillaume Brunerie. *On the homotopy groups of spheres in homotopy type theory.* PhD thesis, Université de Nice Sophia Antipolis, June 2016.

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. https://homotopytypetheory.org/book, Institute for Advanced Study, 2013.