Setoids are not an LCCC

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Abstract

We show that neither the category of setoids nor the category of small groupoids is locally cartesian closed.

1 Preliminaries

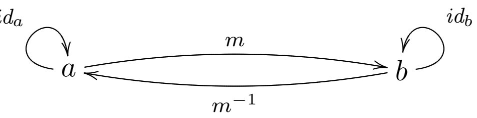
A category is called a *groupoid* if every morphism is an isomorphism. Further, it is *small* if the class of all the morphisms (and hence also the class ob objects) is a set. By **Gpd**, we denote the (large) category of small groupoids. More precisely, the objects ob **Gpd** are the small groupoids and the morphisms are the functors between them. Note that the equality on the hom-sets is *strict* functor equality, while it would also be possible to identify functors if they are naturally isomorphic (and, of course, the small groupoids with functors and natural transformations can also be used to define a 2-category). **Gpd** has a full subcategory that we call **Std**, the category of *setoids*; a small groupoid is an object in this subcategory iff none of its hom-sets contains more than one morphism.

Both **Gpd** and **Std** can be used to define a *CwF*, a *category with families* [2] and therefore defines a model of intensional type theory. Concretely, **Std** has been used by Altenkirch to define an extensional model of type theory [1], while **Gpd** has served Hofmann & Streicher to show that the principle *uniqueness of identity proofs* is not derivable from the equality type eliminator J [3].

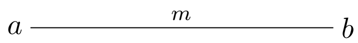
Recall that both **Gpd** and **Std** are *cartesian closed* and *complete*. In these notes, we show that none of them is *locally cartesian closed* by giving a simple counterexample of a morphism the corresponding pullback functor of which does not have a right adjoint. This is well-known in the case of **Gpd** and should also be well-known in the case of **Std**, however, in the latter case, there seems to be some general confusion.

2 The Counterexample

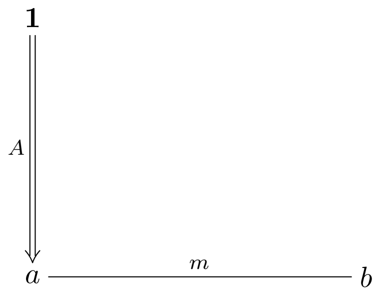
Define **I** to be the "interval" setoid/groupoid, i. e. the groupoid with two objects a, b and one non-identity isomorphism $m : a \to b$ (as well as its inverse):



We choose to keep the inverse and the identities implicitly to increase the readability, so we just picture **I** as



By $\mathbf{1} = \{\star\}$, we denote the trivial setoid/groupoid with exactly on object. Define $A : \mathbf{1} \to \mathbf{I}$ to be the functor that maps the single object to a, as pictured in:



For simplicity, we will now work in **Gpd**, but it will always be clear that the argumentation works for **Std** in exactly the same way.

3 Examining the Pullback Functor

A induces a functor $\Sigma_A : \mathbf{Gpd} \to \mathbf{Gpd/I}$ by post-composition, $\Sigma_A(F) = A \circ F$, where we use that the slice of **Gpd** over **1** is **Gpd** again. This functor Σ_A does have a right adjoint, the *pullback functor*, denoted by $A^\star$. In fact, **Gpd** and **Std** are *complete*. This is easy to see, but we do not need it and we do not even need that $A^\star$ exists, as we are already done if it does not. But actually, it does exist and for any $F : \mathbf{Gpd}$ and $G : \mathbf{Gpd/I}$, we have the hom-set isomorphism

$$\frac{\Sigma_A(F) \Rightarrow_{\mathbf{Gpd/I}} G}{F \Rightarrow_{\mathbf{Gpd}} A^\star(G)} \tag{1}$$

Clearly, [the domain of the functor] $\Sigma_A(F) = A \circ F$ is just F itself, and the functor maps it completely on the "left part" of **I**, i. e. on a. So, any functor

$h : \Sigma_A(F) \Rightarrow_{\mathbf{Gpd/I}} G$ can just be described by explaining how it maps F on the part of G that is in the fibre over a. But now, by the Yoneda lemma, the fibre of G over a is isomorphic to $A^\star(G)$.

4 The non-Existence of Π_A

To show that **Gpd** is not locally cartesian closed, we explain why there cannot be a functor $\Pi_A : \mathbf{Gpd} \to \mathbf{Gpd/I}$ that is right adjoint to $A^\star$. So, assume that Π_A exists. Then, we have (for G as above and a fixed $H : \mathbf{Gpd}$) the hom-set isomorphism

$$\frac{A^\star(G) \Rightarrow_{\mathbf{Gpd}} H}{G \Rightarrow_{\mathbf{Gpd/I}} \Pi_A(H)} \tag{2}$$

Consider the subcategory of all objects $G : \mathbf{Gpd/I}$ such that the fibre over b is empty, i. e. "everything" is mapped to a. Clearly, this subcategory of **Gpd/I** is just [isomorphic to] **Gpd**. In this sense, G is just the same as $A^\star(G)$, as we have seen in the previous section. We can compare the fibre of $\Pi_A(H)$ over a and H as objects in **Gpd**. By the Yoneda lemma, they are isomorphic.

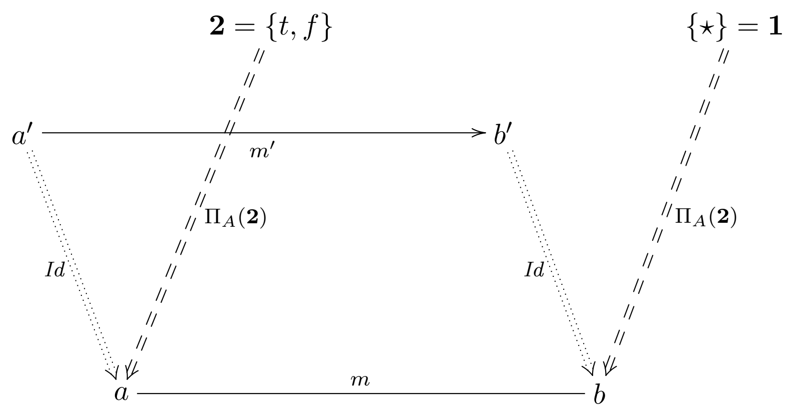
Let us now analyse the fibre of $\Pi_A(H)$ over b. Consider those $G : \mathbf{Gpd/I}$ that have nothing in the fibre over a. Then, $A^\star(G)$ is the empty groupoid, so there is exactly one morphism in the upper hom-set of the hom-set isomorphism (2) and hence exactly one in the lower hom-set. This proves that the fibre of $\Pi_A(H)$ over b is just the terminal object $\mathbf{1}$.

So far, we know how the two fibres of $\Pi_A(H)$ look like:

$$\begin{array}{ccc} H & & \mathbf{1} \\ \Pi_A(H) \Downarrow & & \Downarrow \Pi_A(H) \\ a & \overset{m}{\rule[0.5ex]{6em}{0.4pt}} & b \end{array}$$

What is still left to do is examining the "fibre" over the isomorphism m and here, we get the required contradiction.

For H, choose the discrete groupoid $\mathbf{2}$ which has two objects and no non-identity morphism. We call the two objects t and f (as in *true* and *false*, justified by $\mathbf{2} = \mathbf{Bool}$). For G, chose the identity functor $Id_{\mathbf{I}}$, which clearly is an object in **Gpd/I**. To make it clear, we write $Id_{\mathbf{I}} : \mathbf{I}' \to \mathbf{I}$ and call the parts of $\mathbf{I}'$ just a', b' and m'.



We have $A^\star(Id_{\mathbf{I}}) = \mathbf{1} = \{a'\}$, so the hom-set isomorphism (2) becomes

$$\frac{\mathbf{1} \Rightarrow_{\mathbf{Gpd}} \mathbf{2}}{Id_{\mathbf{I}} \Rightarrow_{\mathbf{Gpd/I}} \Pi_A(\mathbf{2})} \tag{3}$$

Clearly, there are exactly 2 morphisms in the upper hom-set of (3), which shows that there have to be two morphisms in the lower hom-set. This situation is pictured in the figure above. But then, as those two morphisms are functors, the only possibility is one functor T with $T(a') = t$ and one functor F with $F(a') = f$. In both cases, b' has to be mapped on the single element in the fibre over b, so we have $T(b') = F(b')$. But now, T shows that there is some morphism between $t = T(a')$ and $T(b')$, namely $T(m')$, while F shows that there is some morphism between $f = F(a')$ and $F(b')$, namely $T(m')$. Composing these yields a morphism between t and f, living in the fibre over a, which is a contradiction since we know that the fibre over a is the discrete groupoid $\mathbf{2} = \{t, f\}$.

References

[1] Thorsten Altenkirch. "Extensional Equality in Intensional Type Theory". In: *In LICS 99.* IEEE Computer Society Press, 1999, pp. 412–420. URL: `www.cs.nott.ac.uk/~txa/publ/lics99.pdf`.

[2] Martin Hofmann. "Syntax and Semantics of Dependent Types". In: *Semantics and Logics of Computation.* Cambridge University Press, 1997, pp. 79–130. URL: `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.36.8985&rep=rep1&type=pdf`.

[3] Martin Hofmann and Thomas Streicher. "The Groupoid Interpretation of Type Theory". In: *In Venice Festschrift.* Oxford University Press, 1996, pp. 83–111. URL: `www.mathematik.tu-darmstadt.de/~streicher/venedig.ps.gz`.