Connecting Constructive Notions of Ordinals in Homotopy Type Theory

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What are ordinals?

One answer: **Numbers** for counting/ordering:

$$0,\quad 1,\quad 2,\quad 3,\quad \dots,\quad \omega,\quad \omega+1,\quad \omega+2,\quad \dots$$

$$\omega\cdot 2,\quad \omega\cdot 2+1,\quad \dots,\quad \omega^2,\quad \dots,\quad \omega^2\cdot 3+\omega\cdot 7+13,\quad \dots,$$

$$\omega^\omega,\quad \dots,\quad \varepsilon_0=\omega^{\omega^{\omega^{\cdots}}},\quad \dots,\quad \varepsilon_{17},\quad \dots,\quad \omega_1,\quad \dots$$

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- **transitive**: $(a < b) \to (b < c) \to (a < c)$
- **wellfounded**: every sequence $a_0 > a_1 > a_2 > a_3 > \ldots$ terminates
- and **trichotomous**: $(a < b) \vee (a = b) \vee (b < a)$
- ... or **extensional** (instead of trichotomous):
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What are ordinals good for?

CS standard applications:

- proving termination of processes
  (fun example: *Hydra game*)
- justifying recursive definitions /
  (why does the Ackermann
  function terminate?)
- …

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not possible constructively

easy

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Ordinals in dependent type theory

⇒ **Problem/feature** of a constructive setting: different definitions differ!

Three standard notions of "ordinals" in computer science:

- Cantor normal forms
- Brouwer trees
- Wellfounded & extensional & transitive orders

What's the connection? Why can we call them "ordinals"?

Developments in this paper:

(i) axiomatic framework for ordinals and ordinal arithmetic
(ii) "correct" formulation of Brouwer trees (quotient inductive-inductively)
(iii) connections between the three notions and their arithmetic operations

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Motivation: $\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_n}$ with $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$

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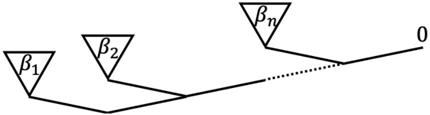
Let $\mathcal{T}$ be the type of *unlabeled binary trees*:

$$\begin{array}{lcl} 0 & : & \mathcal{T} \\ \omega^{-} + - & : & \mathcal{T} \to \mathcal{T} \to \mathcal{T} \end{array}$$

$\alpha =$ 

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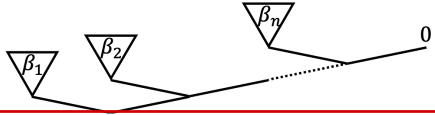
$\alpha =$ 

$\boxed{\text{A tree is a } \textit{Cantor normal form} \text{ if } \beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \text{ (}\textit{lexicographical order}\text{).}}$

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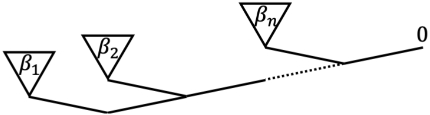
$\alpha =$ 

A tree is a *Cantor normal form* if $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ (*lexicographical order*).

Cannot calculate limits of sequences, but everything else works
– including continuity of arithmetic operations!

Brouwer trees (a.k.a. Brouwer ordinal trees)

How about this inductive type $\mathcal{O}$ of Brouwer trees?

$$\mathsf{zero} : \mathcal{O} \qquad \mathsf{succ} : \mathcal{O} \to \mathcal{O} \qquad \mathsf{sup} : (\mathbb{N} \to \mathcal{O}) \to \mathcal{O}$$

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Problem: $\mathsf{sup}(0, 1, 2, 3, \ldots) \neq \mathsf{sup}(1, 2, 3, \ldots)$

How to fix this without losing wellfoundedness, validity of arithmetic operations, and so on?

Brouwer trees quotient inductive-inductively

(cubical Agda)

```
data Brw where
  zero  : Brw
  succ  : Brw → Brw
  limit : (f : ℕ → Brw) → {f↑ : increasing f} → Brw
  bisim : ∀ f {f↑} g {g↑} →
          f ≈ g →
          limit f {f↑} ≡ limit g {g↑}
  trunc : isSet Brw

data _≤_ where
  ≤-zero      : ∀ {x} → zero ≤ x
  ≤-trans     : ∀ {x y z} → x ≤ y → y ≤ z → x ≤ z
  ≤-succ-mono : ∀ {x y} → x ≤ y → succ x ≤ succ y
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  limit : (f : ℕ → Brw) → {f↑ : increasing f} → Brw
  bisim : ∀ f {f↑} g {g↑} →
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  ≤-zero      : ∀ {x} → zero ≤ x
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Brouwer trees quotient inductive-inductively

(cubical Agda)

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Caveat: Cannot define $a \cdot \mathrm{limit}(f_i) = \mathrm{limit}(a \cdot f_i)$

Not trichotomous ($a < b$ undecidable), everything else works – notably wellfoundedness and arithmetic.

Extensional wellfounded orders

Definition of type Ord:

Pairs $(X : \mathsf{Type}, \prec : X \to X \to \mathsf{Prop})$ such that $\prec$ is transitive, extensional, wellfounded.

$(X, \prec_X) \leq (Y, \prec_Y)$ is given by:

A *monotone* function $f : X \to Y$
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Unsurprisingly, nothing is decidable. E.g. deciding whether $x : \mathsf{Ord}$ is a successor implies LEM (in the HoTT sense).

Connections between the notions

decidable | partially decidable | undecidable

$$\boxed{\mathsf{Cnf}} \xhookrightarrow{\mathsf{CtoB}} \boxed{\mathsf{Brw}} \xhookrightarrow{\mathsf{BtoO}} \boxed{\mathsf{Ord}}$$

$(\omega^a + b) \mapsto \omega^{\mathsf{CtoB}(a)} + \mathsf{CtoB}(b)$

$A \mapsto \Sigma(Y : \mathsf{Brw}).Y < A$

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- preserves and reflects $<$, $\leq$
- commutes with $+$, $*$, ω^x
- bounded (by ϵ_0)

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