Non-Recursive Truncations

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Truncation

\[ \exists(x : A), P(x) \] is not the same as \[ \Sigma(x : A), P(x) \]

\[ := \| \Sigma(x : A), P(x) \| \]

Explanation: The [propositional] truncation \( \| - \| \) makes a type propositional (all elements equal).


<table>
<thead>
<tr>
<th>Rules for ( | - | )</th>
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<tbody>
<tr>
<td>( | A | ) is propositional</td>
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<tr>
<td>( | A | \rightarrow B )</td>
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<td>( A \rightarrow | A | )</td>
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<td>( | A | \rightarrow B \rightarrow )</td>
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<tr>
<td>( A \rightarrow B )</td>
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<td>if ( B ) is propositional</td>
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But then, what is $\|A\| \rightarrow B$?

$\|A\| \rightarrow B$ is equivalent to ...

$\sum(f : A \rightarrow B)$, if $B$ is $(-1)$-type
$\sum(c : \text{wconst}_f)$, if $B$ is $0$-type
$\sum(d : \text{coh}_{f,c})$, if $B$ is $1$-type

... ... ...

general $B$: infinitely many components!

note: $\text{wconst}_f \equiv \prod_{x,y:A} f x = f y$
$\text{coh}_{f,c} \equiv \prod_{x,y,z:A} c(x, y) \cdot c(y, z) = c(x, z)$
...
Theorem [K., TYPES 2014 proceedings]
We can define Reedy fibrant $\mathcal{T}A$ and $\mathcal{E}B : \Delta_+^{op} \to \text{Type}$ such that:

$$\langle \|A\| \to B \rangle \simeq \text{nat. trans. from } \mathcal{T}A \text{ to } \mathcal{E}B$$

in any type theory with $1, \Sigma, \Pi, \text{Id}, \text{fun.ext.}, \|\_\|, \text{Reedy } \omega^{op}-\text{limits.}$

This (directly or indirectly) generalises

* Lurie, *Higher Topos Theory*, Prop. 6.2.3.4: 
  $\infty$-semitopos instead of Type

* Rezk, *Toposes and Homotopy Toposes*, Prop. 7.8: 
  model topos instead of Type
Truncation as a Higher Inductive Type

\[ \|A\| \text{ as HIT} \]
(standard construction)
\[ |\_| : A \to \|A\| \]
\[ t : \prod_{x,y:\|A\|} x = \|A\| y \]

Can we give an equivalent definition of \( \|A\| \) with a nicer elimination principle?

1\(^{st}\) approximation: \( A_1 \)
\[ f : A \to A_1 \]

2\(^{nd}\) approximation: \( A_2 \)
\[ f : A \to A_2 \]
\[ c : w\text{const}_f \]

3\(^{rd}\) approximation: \( A_3 \)
\[ f : A \to A_3 \]
\[ c : w\text{const}_f \]
\[ d : \text{coh}_{f,c} \]

\[ \|A\| \simeq \|A_1\|_0 \simeq \|A_2\|_1 \simeq \ldots \]
Easier elimination principle into 0-, or 1-, or \ldots-types!
Purely non-recursive representations, I

We could try to consider the homotopy colimit of

\[ A_1 \to A_2 \to A_3 \to \ldots \]

which should be \( \| A \| \).

Problem: for any \( n \), we can write down \( A_n \). However, we cannot write down \( A : \mathbb{N} \to U \).

(“Semisimplicial Types Phenomenon”)
Purely non-recursive representations, II

Solution: Make the sequence $A_1 \to A_2 \to A_3 \to \ldots$ “coarser”.

* van Doorn (CPP’16), independent of my analysis: do the first approximation in every step (easy to prove correct, but no finite special cases).

* K. (LiCS’16): construct $A_{n+1}$ by taking $A_n$ and adding fillers for $S^{n-1} \to A_n$ (harder to prove correct, but useful finite special cases);

Any sequence of weakly constant functions has a propositional colimit!

* Rijke - van Doorn / Buchholtz - Rijke, wip: localizations and related constructions

Thank you! Any comments or questions?