Induction for Cycles

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Types in Munich ’20
(online substitution thereof)
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based on arxiv.org/abs/2001.07655
Consider paths in a graph.

If we want to prove a property…

– *for all paths*: Induction!
– *for all closed paths*: how???
Consider paths in a graph.

If we want to prove a property…

− for all paths: **Induction**!
− for all closed paths: **how??**

Aim of this project:
approach for a special case
+ applications in HoTT.
Quotients in Type Theory (Hofmann)

Given: \( A : \text{Set} \)
\( \sim : A \to A \to \text{Set} \)

We get: \( A/\sim : \text{Set} \)

Property: for \( B : \text{Set} \),

\[
\begin{align*}
  f : A & \to B \\
  h : (a_1 \sim a_2) & \to f(a_1) = f(a_2) \\
  g : (A/\sim) & \to B
\end{align*}
\]
Quotients in Type Theory (Hofmann)

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Property: for \( B : \text{Set} \),
\[ f : A \to B \]
\[ h : (a_1 \sim a_2) \to f(a_1) = f(a_2) \]
\[ \sim \]
\[ g : (A/\sim) \to B \]

In homotopy type theory:
All of this is for sets
(aka 0-truncated types, types satisfying UIP),
"set-quotients"

What if \( B \) is only
1-truncated
(e.g. the universe of sets)?
Set-Quotients in HoTT

Given: \( A : \text{Type} \)
\( \sim : A \to A \to \text{Type} \)

We get: \( A/\sim : \text{Set} \)

Property: for \( B : 1\text{-Type}, \)

\[
\begin{align*}
    f & : A \to B \\
    h : (a_1 \sim a_2) & \to f(a_1) = f(a_2) \\
    c : (p : a \sim^s a) & \to h^s(p) = \text{refl}_{f(a)}
\end{align*}
\]

\( g : (A/\sim) \to B \)
Set-Quotients in HoTT

Given: $A : \text{Type}$

$\sim : A \to A \to \text{Type}$

We get: $A/\sim : \text{Set}$

Property: for $B : 1$-$\text{Type}$,

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\begin{align*}
 f & : A \to B \\
 h & : (a_1 \sim a_2) \to f(a_1) = f(a_2) \\
 c & : (p : a \sim^{s*} a) \to h^{s*}(p) = \text{refl}_{f(a)}
\end{align*}
\]

$g : (A/\sim) \to B$

The cycle $p$ in $A$:

\[
\begin{array}{ccccccc}
 & & a_3 & & & & a_4 \\
 p_3 & & & & & & \downarrow \\
 a_2 & & & & & & a_4 \\
 & \uparrow & & & & & \\
 & p_2 & & & & & p_0 \\
 a_1 & & & & & & a_0 \\
 & \downarrow & & & & & p_1 \\
 & a_0 & & & & & a_1 \\
 & \downarrow & & & & & \downarrow \\
 & f(a_0) & & & & & f(a_1) \\
 & h(p_1) & & & & & h(p_1)^{-1}
\end{array}
\]

Its image in $B$:

\[
\begin{align*}
 h(p_3)^{-1} & \quad f(a_3) & \quad h(p_4) \\
 f(a_2) & \quad c(p) & \quad f(a_4) \\
 h(p_2) & \quad h(p_5) & \\
 f(a_1) & \quad f(a_0)
\end{align*}
\]
Set-Quotients in HoTT

Given:

\[ A : \text{Type} \]
\[ \sim : A \to A \to \text{Type} \]

We get:

\[ A/\sim : \text{Set} \]

Instance of the general problem!!

\[ f : A \to B \]
\[ h : (a_1 \sim a_2) \to f(a_1) = f(a_2) \]
\[ c : (p : a \sim^* a) \to h^*(p) = \text{refl}_{f(a)} \]

\[ g : (A/\sim) \to B \]

The cycle \( p \) in \( A \):

\[ a_0 \xrightarrow{p_0} a_1 \xrightarrow{p_1} a_2 \xrightarrow{p_2} a_3 \xrightarrow{p_3} a_4 \]

Its image in \( B \):

\[ f(a_0) \]
\[ f(a_1) \]
\[ f(a_2) \]
\[ f(a_3) \]
\[ f(a_4) \]

\[ h(p_1)^{-1} \]
\[ h(p_2) \]
\[ h(p_3)^{-1} \]
\[ h(p_4) \]
\[ h(p_5) \]
An Example in HoTT

Given: $M : \text{Set}$
Want: $\text{Free Group on } M$

In Sets (ordinary free group):

Set-quotient $\text{List}(M + M)/\sim$

$[x_0, \ldots, x_{k-1}, x_k, x_k^{-1}, x_{k+1}, \ldots, x_n]$

$\sim$

$[x_0 \ldots, x_{k-1}, x_{k+1}, \ldots, x_n]$
An Example in HoTT

Given: \( M : \text{Set} \)

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\[ \sim \]

\[ [x_0 \ldots, x_{k-1}, x_{k+1}, \ldots, x_n] \]

Higher-categorical free group:

\( \Omega(\text{hcolim}(M \Rightarrow 1)) \)
An Example in HoTT

Given: \( M \) : Set

Want: Free Group on \( M \)

In Sets (ordinary free group):

Set-quotient $\text{List}(M + M)/\sim$

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\begin{align*}
[x_0, \ldots, x_{k-1}, x_k, x_k^{-1}, x_{k+1}, \ldots, x_n] \\
\sim \\
[x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n]
\end{align*}
\]

Higher-categorical free group:

\( \Omega(\text{hcolim}(M \Rightarrow 1)) \)

\( H \equiv \text{hcolim}(M \Rightarrow 1) \)

can be implemented as a higher inductive type:

inductive \( H \)

base : \( H \)

loops : \( M \to \text{base} = \text{base} \)
Free Groups

Sets
(1-category)

FreeGroups
(1-category)

List\( (M + M)/\sim \) \cong \Omega(\text{hcolim}(M \Rightarrow 1))

Types
((\infty, 1\)-category)

Free \infty\text{-Groups}
((\infty, 1\)-category)

Yes, with excluded middle.
Unknown (conjecture: independent) otherwise.
Free Groups

\[
\text{List}(M + M)/\sim \simeq \Omega(\text{hcolim}(M \Rightarrow 1))
\]

Needed: map from set-quotient into (a priori) higher type!
List$(M + M)/\sim \simeq \Omega(hcolim(M \Rightarrow 1))$

Needed: map from set-quotient into (a priori) higher type!
First approximation: Does $\Omega(hcolim(M \Rightarrow 1))$ have trivial fundamental groups? ($\sim \|\Omega(hcolim(M \Rightarrow 1))\|_1$)
What would we need?

Recall: \( \text{List}(M + M)/\sim \rightarrow \|\Omega(\text{hcolim}(M \Rightarrow 1))\|_1 \)

is given by:

\[
f : \text{List}(M + M) \rightarrow \|\Omega(\text{hcolim}(M \Rightarrow 1))\|_1
\]

\[
h : (\ell_1 \sim \ell_2) \rightarrow f(\ell_1) = f(\ell_2)
\]

\[
c : h \text{ maps every closed zig-zag to reflexivity}
\]
What would we need?

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h : (\ell_1 \sim \ell_2) \to f(\ell_1) = f(\ell_2)
\]
\[
c : h \text{ maps every closed zig-zag to reflexivity}
\]

easy parts:
\[
f([+m_0, -m_1, +m_2]) \equiv \text{loops}(m_0) \cdot \text{loops}(m_1)^{-1} \cdot \text{loops}(m_2)
\]
\[
h : (\text{use that inverses cancel, recall the def of } \sim:
\[
[+m_0, -m_1, +m_1, +m_2, -m_3] \sim [+m_0, +m_2, -m_3])
\]
What would we need?

Recall: \( \text{List}(M + M) / \sim \rightarrow ||\Omega(hcolim(M \Rightarrow 1))||_1 \)

is given by: \[ f: \text{List}(M + M) \rightarrow ||\Omega(hcolim(M \Rightarrow 1))||_1 \]
\[ h: (\ell_1 \sim \ell_2) \rightarrow f(\ell_1) = f(\ell_2) \]
\[ c: h \text{ maps every closed zig-zag to reflexivity} \]

easy parts:
\[ f([+m_0, -m_1, +m_2]) \equiv \text{loops}(m_0) \cdot \text{loops}(m_1)^{-1} \cdot \text{loops}(m_2) \]
\[ h: (\text{use that inverses cancel, recall the def of } \sim:) \]
\[ [+m_0, -m_1, +m_1, +m_2, -m_3] \sim [+m_0, +m_2, -m_3] \]
\[ c: (\text{should be true, but how to prove it?}) \]
Problem: Prove a property \textit{for every cycle} in a graph.

Assumption: The graph is given by the symmetric closure of a relation
Problem: Prove a property for every cycle in a graph.

Assumption: The graph is given by the symmetric closure of a relation which is:
- locally confluent
- Noetherian (co-wellfounded).

\[
\begin{align*}
&\begin{bmatrix}
-m_0, +m_1, -m_1, -m_2, -m_3, +m_3 \\
-m_0, -m_2, -m_3, +m_3 \\
-m_0, m_1, -m_1, -m_2 \\
-m_0, -m_2
\end{bmatrix} \\
&\begin{bmatrix}
-m_0, +m_1, -m_1, -m_2, -m_3, +m_3 \\
-m_0, -m_2, -m_3, +m_3 \\
-m_0, m_1, -m_1, -m_2 \\
-m_0, -m_2
\end{bmatrix}
\end{align*}
\]
Problem: Prove a property for every cycle in a graph.

Assumption: The graph is given by the symmetric closure of a relation which is:
- locally confluent
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Our proposed solution:
1. Given a relation $\rightsquigarrow$ on $A$, we define a new relation $\rightsquigarrow^{\circ}$ on cycles $a \rightsquigarrow^{s*} a$.
2. If $\rightsquigarrow$ is Noetherian, then so is $\rightsquigarrow^{\circ}$.
3. If $\rightsquigarrow$ further is locally confluent, then any cycle can be split into a $\rightsquigarrow^{\circ}$-smaller cycle and a confluence cycle

$\Rightarrow$ Induction is possible!
**Definition.** Let $\rightsquigarrow$ be a relation on $A$. Then, $\rightsquigarrow^L$ on $\text{List}(A)$ is generated by

$$[\vec{a}_1, a, \vec{a}_2] \rightsquigarrow^L [\vec{a}_1, x_0, x_1, \ldots, x_k, \vec{a}_2]$$

for all $x_i$ with $a \rightsquigarrow x_i$. 

**Lemma.** $(\rightsquigarrow^L \text{Noetherian}) \Rightarrow (\rightsquigarrow \text{Noetherian})$.

**Proof.**
1. If $\ell_1$ and $\ell_2$ are both $\rightsquigarrow^L$-accessible, then so is $\ell_1 + \ell_2$. (Proof: by double "accessibility induction").
2. If $a : A$ is $\rightsquigarrow$-accessible, then $[a]$ is $\rightsquigarrow^L$-accessible. (Proof: $[a] \rightsquigarrow^L [x_0, \ldots, x_k]$; induction hypothesis + above.)
3. If every $a_i$ is $\rightsquigarrow$-accessible, then $[a_0, \ldots, a_n]$ is $\rightsquigarrow^L$-accessible. (Proof: first point.)
Step 1

**Definition.** Let $\sim$ be a relation on $A$. Then, $\sim^L$ on List$(A)$ is generated by
\[
[\vec{a}_1, a, \vec{a}_2] \sim^L [\vec{a}_1, x_0, x_1, \ldots, x_k, \vec{a}_2]
\]
for all $x_i$ with $a \sim x_i$.

**Lemma.** $(\sim$ Noetherian) $\Rightarrow$ $(\sim^L$ Noetherian).
Step 1

**Definition.** Let $\leadsto$ be a relation on $A$. Then, $\leadsto^L$ on $\text{List}(A)$ is generated by

$$[\vec{a}_1, a, \vec{a}_2] \leadsto^L [\vec{a}_1, x_0, x_1, \ldots, x_k, \vec{a}_2]$$

for all $x_i$ with $a \leadsto x_i$.

**Lemma.** ($\leadsto$ Noetherian) $\Rightarrow$ ($\leadsto^L$ Noetherian).

Proof.

1. If $\ell_1$ and $\ell_2$ are both $\leadsto^L$-accessible, then so is $\ell_1 + \ell_2$. (Proof: by double “accessibility induction”.)
Step 1

**Definition.** Let $\rightsquigarrow$ be a relation on $A$. Then, $\rightsquigarrow^L$ on $\text{List}(A)$ is generated by

$$[\vec{a}_1, a, \vec{a}_2] \rightsquigarrow^L [\vec{a}_1, x_0, x_1, \ldots, x_k, \vec{a}_2]$$

for all $x_i$ with $a \rightsquigarrow x_i$.

**Lemma.** ($\rightsquigarrow$ Noetherian) $\Rightarrow$ ($\rightsquigarrow^L$ Noetherian).

**Proof.**

1. If $\ell_1$ and $\ell_2$ are both $\rightsquigarrow^L$-accessible, then so is $\ell_1 + \ell_2$. (Proof: by double “accessibility induction”.)

2. If $a : A$ is $\rightsquigarrow$-accessible, then $[a]$ is $\rightsquigarrow^L$-accessible. (Proof: $[a] \rightsquigarrow^L [x_0, \ldots, x_k]$; induction hypothesis + above.)
Step 1

**Definition.** Let \( \rightsquigarrow \) be a relation on \( A \).
Then, \( \rightsquigarrow^L \) on List\((A)\) is generated by
\[
[\vec{a}_1, a, \vec{a}_2] \rightsquigarrow^L [\vec{a}_1, x_0, x_1, \ldots, x_k, \vec{a}_2]
\]
for all \( x_i \) with \( a \rightsquigarrow x_i \).

**Lemma.** \((\rightsquigarrow \text{ Noetherian}) \implies (\rightsquigarrow^L \text{ Noetherian})\).

**Proof.**

1. If \( \ell_1 \) and \( \ell_2 \) are both \( \rightsquigarrow^L \)-accessible, then so is \( \ell_1 + \ell_2 \).
   (Proof: by double “accessibility induction”.)

2. If \( a : A \) is \( \rightsquigarrow \)-accessible, then \([a]\) is \( \rightsquigarrow^L \)-accessible. (Proof: \([a] \rightsquigarrow^L [x_0, \ldots, x_k]; \) induction hypothesis + above.)

3. If every \( a_i \) is \( \rightsquigarrow \)-accessible, then \([a_0, \ldots, a_n]\) is \( \rightsquigarrow^L \)-accessible. (Proof: first point.)
Lemma. (⇝ Noetherian) ⇒
(any cycle is either empty or contains a span).

Span: \( a' \leftrightarrow a \rightsquigarrow a'' \)
**Lemma.** ($\rightsquigarrow$ Noetherian) $\Rightarrow$
(any cycle is either empty or contains a span).

**Definition.** For $\gamma$ a cycle, write $\varphi(\gamma)$ for the vertex sequence of $\gamma$.
Write $\gamma \rightsquigarrow^\circ \delta$ if $\varphi(\gamma) \rightsquigarrow^L \varphi(\delta')$ for any rotation $\delta'$ of $\delta$. 

Span : $a' \rightsquigarrow a \rightsquigarrow a''$
Step 2

Lemma. ($\rightsquigarrow$ Noetherian) $\Rightarrow$
(any cycle is either empty or contains a span).

Definition. For $\gamma$ a cycle, write $\varphi(\gamma)$ for the vertex sequence of $\gamma$.
Write $\gamma \rightsquigarrow^\circ \delta$ if $\varphi(\gamma) \rightsquigarrow^L \varphi(\delta')$ for any rotation $\delta'$ of $\delta$.

Lemma. ($\rightsquigarrow$ Noetherian) $\Rightarrow$ ($\rightsquigarrow^{+\circ+}$ Noetherian).
Theorem. ($\rightsquigarrow$ Noetherian and locally confluent) $\Rightarrow$
(Any cycle can be written as the “merge” of a $\rightsquigarrow^{\circ+}$-smaller cycle and a confluence cycle).
Theorem. (\(\rightsquigarrow\) Noetherian and locally confluent) \(\Rightarrow\)
(any cycle can be written as the “merge” of a \(\rightsquigarrow^+\)-smaller cycle and a confluence cycle).
Theorem. (⇝ Noetherian and locally confluent) ⇒
(any cycle can be written as the “merge” of a ⇝⁺⁺-smaller cycle and a confluence cycle).

\[ \begin{align*}
&\quad a_0 \\
&\quad a_1 \quad a_2 \\
&\quad a_3 \quad a_4 \quad a_5 \\
&\quad a_6 \quad a_7 \quad a_8 \\
&\quad a_9 \\
\end{align*} \]
Theorem (Noetherian Cycle Induction).
Given: \( A : \text{Type} \)
\[ (\rightsquigarrow) : A \to A \to \text{Type} \]
\( P : \text{cycles} \to \text{Type} \).
Assume further:
- relation \( \rightsquigarrow \) Noetherian and locally confluent
- \( P \) stable under rotating of cycles:
  \[ P(\gamma) \to P(\text{some rotation of } \gamma) \]
- \( P \) stable under “merging” of cycles:
  \[ P(\alpha) \to P(\beta) \to P(\alpha + \gamma) \]
Then: \( P(\text{empty}) \) and \( P(\text{confluence cycle}) \) \( \Rightarrow \) \( P(\text{any cycle}) \).
Theorem (Noetherian Cycle Induction).

Given: \( A : \text{Type} \)
\[
(\leadsto) : A \rightarrow A \rightarrow \text{Type}
\]
\( P : \text{cycles} \rightarrow \text{Type}. \)

Assume further:
- relation \( \leadsto \) Noetherian and locally confluent
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- \( P \) stable under “merging” of cycles:
  \[ P(\alpha) \rightarrow P(\beta) \rightarrow P(\alpha + \gamma) \]

Then: \( P(\text{empty}) \) and \( P(\text{confluence cycle}) \) \( \Rightarrow \) \( P(\text{any cycle}) \).

These conditions are easily checked in our HoTT-examples, where \( P(\gamma) :\equiv \text{the cycle } \gamma \text{ is mapped to a trivial equality.} \)

Can show approximations to other open questions in HoTT with this.


Formalised in Lean (great job by Jakob!).

Thanks!