Type Theory with Weak J

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TYPES, Budapest, 1 June 2017
Report of a discussion between:

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Equalities

$$(\lambda x.x)y \equiv y$$

(a judgment)

$n + 4 = 4 + n$

(a type)

Which equalities do we want to be judgmental/definitional?

Consequences?

Can we prove more stuff if more equalities are judgmental?

E.g.: If we have equality reflection

$x = y \Rightarrow x \equiv y$ ("extensional MLTT"), we can:

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prove function extensionality.
Equalities

\[(\lambda x.x)y \equiv y\]  \[n + 4 = 4 + n\]

(a judgment)  (a type)

Which equalities do we want to be judgmental/definitional? Consequences?
Equalities

\[(\lambda x.x)y \equiv y\]  \hspace{1cm} n + 4 = 4 + n

(a judgment)  \hspace{1cm} (a type)

Which equalities do we want to be judgmental/definitional? Consequences?

Can we prove more stuff if more equalities are judgmental?
Equalities

\[(\lambda x . x)y \equiv y\]  \hspace{1cm} n + 4 = 4 + n

(a judgment) \hspace{1cm} (a type)

Which equalities do we want to be judgmental/definitional? Consequences?

Can we prove more stuff if more equalities are judgmental?

E.g.: If we have equality reflection \(x \equiv y\) ("extensional MLTT"), we can:

- derive UIP/K: \((x : A) \to (p : x = x) \to (p = \text{refl})\),
Equalities

\[(\lambda x.x)y \equiv y\]  \hspace{1cm} n + 4 = 4 + n

(a judgment)  \hspace{1cm} (a type)

Which equalities do we want to be judgmental/definitional? Consequences?

Can we prove more stuff if more equalities are judgmental?

E.g.: If we have equality reflection \(x = y\) \(\quad (\text{"extensional MLTT"})\), we can:

\[
\begin{align*}
\text{derive UIP/K: } & (x : A) \rightarrow (p : x = x) \rightarrow (p = \text{refl}), \text{ because} \\
&(x, y : A) \rightarrow (p : x = y) \rightarrow p = \text{refl} \quad \text{does now type-check.}
\end{align*}
\]
Equalities

\[(\lambda x.x)y \equiv y\]  \[n + 4 = 4 + n\]
(a judgment)  (a type)

Which equalities do we want to be judgmental/definitional?
Consequences?
Can we prove more stuff if more equalities are judgmental?

E.g.: If we have equality reflection \(x = y\) \(\frac{x = y}{x \equiv y}\) (“extensional MLTT”), we can:

- derive UIP/K: \((x : A) \rightarrow (p : x = x) \rightarrow (p = \text{refl})\), because \((x, y : A) \rightarrow (p : x = y) \rightarrow p = \text{refl}\) does now type-check.
- prove function extensionality.
Conservativity


If:

- $A$ is a type in intensional MLTT with funext and UIP
- $A$ is inhabited in extensional MLTT

Then:

- $A$ is inhabited in intensional MLTT with funext and UIP
Conservativity


If:

- $A$ is a type in intensional MLTT with funext and UIP
- $A$ is inhabited in extensional MLTT

Then:

- $A$ is inhabited in intensional MLTT with funext and UIP

Our setting: Intensional MLTT with funext (+ univalence + ...). What happens if we remove/add judgmental equalities?
Write $I_A$ for $\sum_{x,y:A} x = y$.

The type of the equality eliminator is:

$$J : (A : \mathcal{U}) \rightarrow (P : I_A \rightarrow \mathcal{U}) \rightarrow (d : (x : A) \rightarrow P(x, x, \text{refl})) \rightarrow (q : I_A) \rightarrow P(q).$$
Write $I_A$ for $\Sigma_{x,y:A} x = y$.

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The usual judgmental $\beta$-rule says $J^{A,P,d}(x, x, \text{refl}) \equiv d(x)$. 
Weak J

Write $I_A$ for $\sum_{x,y:A} x = y$.
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The usual judgmental $\beta$-rule says $J^{A,P,d}(x, x, \text{refl}) \equiv d(x)$.
What happens if we replace it by

$$J_{\beta} : (A : \mathcal{U}) \to (P : I_A \to \mathcal{U}) \to (d : (x : A) \to P(x, x, \text{refl})) \to (x : A) \to J^{A,P,d}(x, x, \text{refl}) = d(x)$$

(“weak J”) - do we lack coherence?
Example: subst

Recall: Given

\[ A : \mathcal{U} \quad P : A \to \mathcal{U} \quad x, y : A \quad p : x = y \]

we have

\[ \text{subst}^{A,P,p} : P(x) \to P(y). \]
Example: subst

Recall: Given

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we have

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Usually, we have \[ \text{subst}^{A, P, \text{refl}}(q) \equiv q. \]
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we have

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Usually, we have \( \text{subst}^{A,P,\text{refl}}(q) \equiv q \).

From “weak J”, we can only derive

\[ \text{subst}_{\beta}^{A,P} : (q : P(x)) \to \text{subst}^{A,P,\text{refl}}(q) = q. \]
Example: subst

$$\text{ap}_{\text{subst}^{A,P,\text{refl}}}(\text{subst}_{\beta}^{A,P}(q)) \quad \text{subst}_{\beta}^{A,P}(\text{subst}^{A,P,\text{refl}}(q))$$

$$\text{subst}^{A,P,\text{refl}}(q)$$
Example: subst

\[
\text{subst}^{A, P, \text{refl}} \left( \text{subst}^{A, P, \text{refl}}(q) \right)
\]

\[
\text{ap}_{\text{subst}^{A, P, \text{refl}}} \left( \text{subst}^{A, P}_{\beta}(q) \right) = \text{subst}^{A, P}_{\beta} \left( \text{subst}^{A, P, \text{refl}}(q) \right)
\]

\[
\left( \text{subst}^{A, P, \text{refl}}, \text{subst}^{A, P}_{\beta} \right) : \Sigma f : P(x) \to P(x) \left( (q : P(x)) \to f(q) = q \right)
\]
Example: subst

\[ \text{subst}^{A,P,\text{refl}} \left( \text{subst}^{A,P,\text{refl}}(q) \right) \]

\[ \text{ap}_{\text{subst}^{A,P,\text{refl}}} \left( \text{subst}_{\beta}^{A,P}(q) \right) \]

\[ \text{subst}_{\beta}^{A,P} \left( \text{subst}^{A,P,\text{refl}}(q) \right) \]

\[ \text{subst}^{A,P,\text{refl}}(q) \]

\[ \left( \text{subst}^{A,P,\text{refl}}, \text{subst}_{\beta}^{A,P} \right) : \Sigma f:P(x)\rightarrow P(x) ((q : P(x)) \rightarrow f(q) = q) \]

And: \[ \left( \text{subst}^{A,P,\text{refl}}, \text{subst}_{\beta}^{A,P} \right) = (\text{id}_{P(x)}, \lambda q.\text{refl}) \]
Example: $\text{subst}$

$$
\begin{array}{c}
\text{id (id}(q)\text{))}
\end{array}
$$

$$
\begin{array}{c}
\text{ap}_{\text{id}}\text{(refl)}
\end{array}
$$

$$
\begin{array}{c}
\text{refl}
\end{array}
$$

$$
\begin{array}{c}
\text{id}(q)
\end{array}
$$

$$(\text{subst}^{A,P,\text{refl}}, \text{subst}^{A,P}) : \Sigma f : P(x) \to P(x) \left( (q : P(x)) \to f(q) = q \right)$$

And: $$(\text{subst}^{A,P,\text{refl}}, \text{subst}^{A,P}) = (\text{id}_{P(x)}, \lambda q.\text{refl})$$
Weak J revisited

Write $I_A$ for $\Sigma_{x,y:A} x = y$.

The types of J and its (weak) $\beta$-rule are:

\[
\begin{align*}
J : & \quad (A : U) \to (P : I_A \to U) \to (d : (x : A) \to P(x, x, \text{refl})) \\
& \to (q : I_A) \to P(q)
\end{align*}
\]

\[
\begin{align*}
J_\beta : & \quad (A : U) \to (P : I_A \to U) \to (d : (x : A) \to P(x, x, \text{refl})) \\
& \to (x : A) \to J^{A,P,d}(x, x, \text{refl}) = d(x)
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Weak J revisited

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J : \quad (A : \mathcal{U}) \rightarrow (P : I_A \rightarrow \mathcal{U}) \rightarrow (d : (x : A) \rightarrow P(x, x, \text{refl})) \\
\rightarrow (q : I_A) \rightarrow P(q)
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\rightarrow (x : A) \rightarrow J^{A,P,d}(x, x, \text{refl}) = d(x)
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$A \rightarrow I_A, \quad x \mapsto (x, x, \text{refl})$ is an equivalence.
Weak J revisited

Write $I_A$ for $\sum_{x,y:A} x = y$.

The types of $J$ and its (weak) $\beta$-rule are:

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\to (x : A) \to P(x, x, \text{refl})
\]

\[
J_\beta : (A : \mathcal{U}) \to (P : I_A \to \mathcal{U}) \to (d : (x : A) \to P(x, x, \text{refl})) \\
\to (x : A) \to J^{A,P,d}(x) = d(x)
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Write $I_A$ for $\sum_{x,y:A} x = y$.

The types of J and its (weak) $\beta$-rule are:

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J : \quad (A : U) \to (P : I_A \to U) \to (d : (x : A) \to P(x, x, \text{refl})) \to (x : A) \to P(x, x, \text{refl})
\]

\[
J_\beta : \quad (A : U) \to (P : I_A \to U) \to (d : (x : A) \to P(x, x, \text{refl})) \to (x : A) \to J^{A,P,d}(x) = d(x)
\]

$A \to I_A, \quad x \mapsto (x, x, \text{refl})$ is an equivalence.

Conjecture: “Normal” MLTT is conservative over MLTT with weak J.

Thank you!