Type Theory with Weak J

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TYPES, Budapest, 1 June 2017

Report of a discussion between:

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Equalities

$(\lambda x.x)y \equiv y$

(a judgment)

$n + 4 = 4 + n$

(a type)

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Can we prove more stuff if more equalities are judgmental?

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E.g.: If we have equality reflection $\dfrac{x = y}{x \equiv y}$ ("extensional MLTT"), we can:

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- prove function extensionality.

Conservativity

Hofmann 1995 (cf. Oury 2005)

If:

- A is a type in intensional MLTT with funext and UIP
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Our setting: Intensional MLTT with funext (+ univalence + ...). What happens if we remove/add judgmental equalities?

Weak J

Write I_A for $\Sigma_{x,y:A} x = y$.

The type of the equality eliminator is:

$$\mathsf{J} : \quad (A : \mathcal{U}) \to (P : I_A \to \mathcal{U}) \to (d : (x : A) \to P(x, x, \mathsf{refl})) \to (q : I_A) \to P(q).$$

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What happens if we replace it by

$$\begin{aligned} \mathsf{J}_\beta : \quad & (A : \mathcal{U}) \to (P : I_A \to \mathcal{U}) \to (d : (x : A) \to P(x, x, \mathsf{refl})) \\ & \to (x : A) \to \mathsf{J}^{A,P,d}(x, x, \mathsf{refl}) = d(x) \end{aligned}$$

("weak J") - do we lack coherence?

Example: subst

Recall: Given

$$A : \mathcal{U} \qquad P : A \to \mathcal{U} \qquad x, y : A \qquad p : x = y$$

we have

$$\mathsf{subst}^{A,P,p} : P(x) \to P(y).$$

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From "weak J", we can only derive

$$\mathsf{subst}_{\beta}^{A,P} : (q : P(x)) \to \mathsf{subst}^{A,P,\mathsf{refl}}(q) = q.$$

Example: subst

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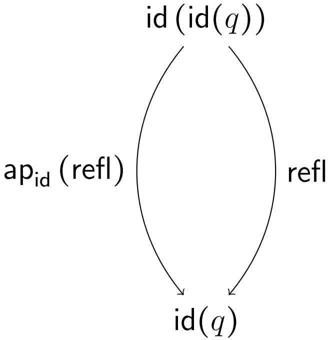

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Weak J revisited

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Conjecture: "Normal" MLTT is conservative over MLTT with weak J.

Thank you!