Some connections between open problems

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Open Problems

Define semi-simplicial types

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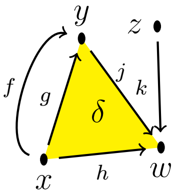
SEMISIMPLICIAL TYPES

(UF 2012/13, Lumsdaine et al.)

$$A_0 : \mathcal{U}$$
$$A_1 : A_0 \times A_0 \to \mathcal{U}$$
$$A_2 : \Pi(x, y, z : A_0), A_1(x, y) \times A_1(y, z) \times A_1(x, z) \to \mathcal{U}$$



Example:

$A_0 \equiv \{x, y, z, w\}$
$A_1(x, y) \equiv \{f, g\}$
$A_1(x, w) \equiv \{h\}$,
...
$A_2(.., g, j, h) \equiv \{\delta\}$

Open Problems

Define semi-simplicial types

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\begin{aligned}
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PROBLEM: Find $\mathbf{F} : \mathbb{N} \to \mathcal{U}_\mathbf{1}$ such that $\mathbf{F}(\mathbf{n}) \simeq$ type of tuples $(\mathbf{A_0}, \ldots, \mathbf{A_n})$.

Open Problems

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UNSOLVED in "book-HoTT", solved in Voevodsky's HTS, our 2LTT (arXiv:1705.03307)

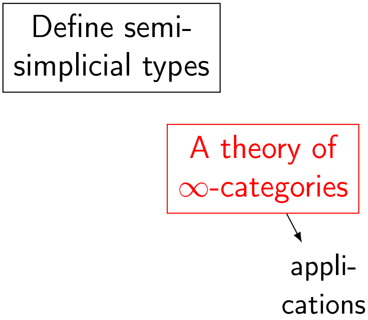
NOTE: ***semi*** is what allows the above encoding.

Open Problems

Define semi-simplicial types

A theory of ∞-categories

Open Problems

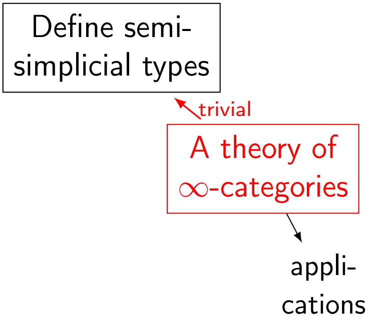


HIGHER CATEGORIES

TASK: develop $(\infty, 1)$-cat's *in* HoTT (internally)

WHY: occur everywhere in HoTT, e.g. "HIT H is initial in the category of H-algebras" (not captured by AKS)

Open Problems



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NOTE: "Semisimplicial types" $\approx$ functors $\Delta_+^{\mathsf{op}} \to \mathcal{U}$ ("simpl. types" $\approx \Delta^{\mathsf{op}} \to \mathcal{U}$, others see K.-Sattler'17).

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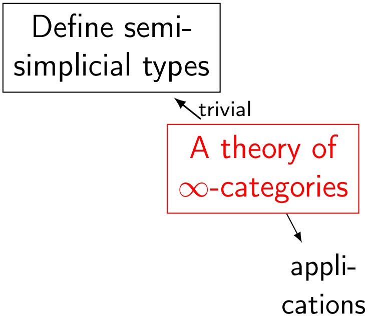

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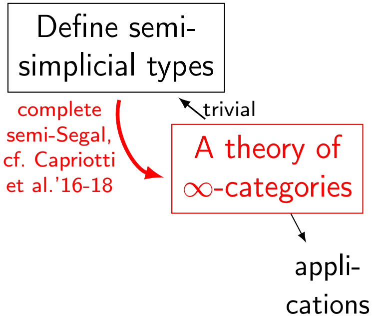
APPROACH (if given SST): mimic Rezk's *Segal spaces* (replace *space* by *type*); issue: ***semi***

Open Problems



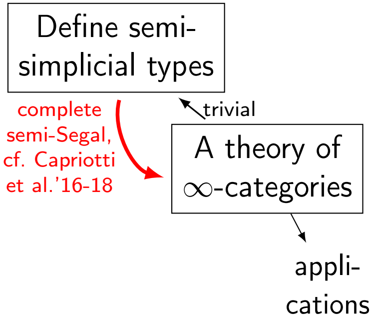
COMPLETE SEMI-SEGAL TYPE

Def:

- semisimpl. type $(A_0, A_1, \ldots)$
- Segal cond. (aka horn filling)
- completeness (Harpaz'15)

(note: Segal, compl. are prop)

Open Problems



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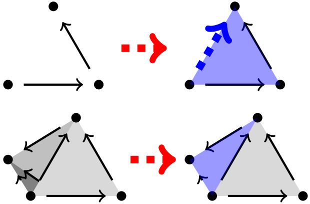
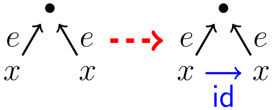


(comp.)

(assoc.)

Completeness:



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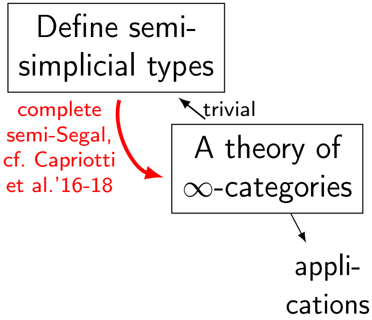

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Special case:

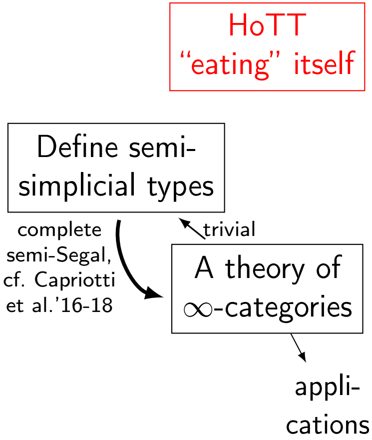
- A_1 family of sets

($\Rightarrow$ only need $A_0, \ldots, A_3$)

$\simeq$ AKS categories

(also works one level higher). Capriotti-K.'18

Open Problems



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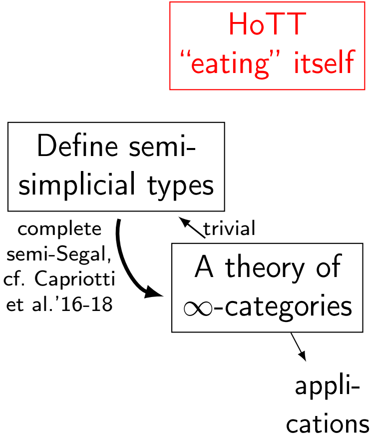
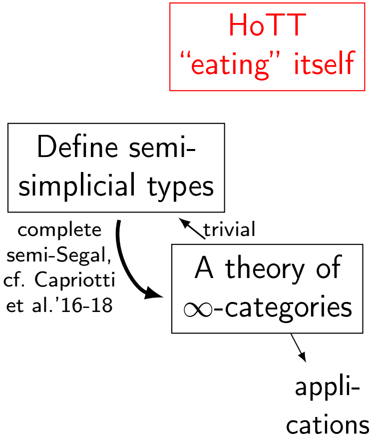


TYPE THEORY IN TYPE THEORY

QUESTION (SHULMAN'14): Does HoTT with (n+1) universes model HoTT with n universes?

Can we implement HoTT in HoTT?

Open Problems



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Some partial results, e.g. Altenkirch-Kaposi's *type theory in type theory*, Escardó, Xu, Buchholtz, Lumsdaine, Weaver, Tsementzis, ...

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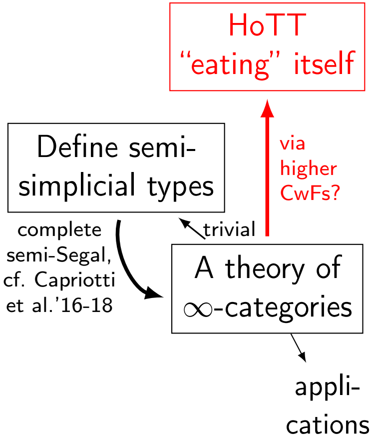

PROBLEM:

- syntax should be a set
- set-truncate $\Rightarrow$ HoTT not a model
- idea: use ∞-cat’s to add all coherences (cf. Altenkirch)

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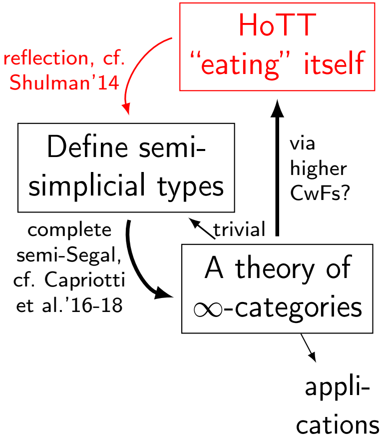

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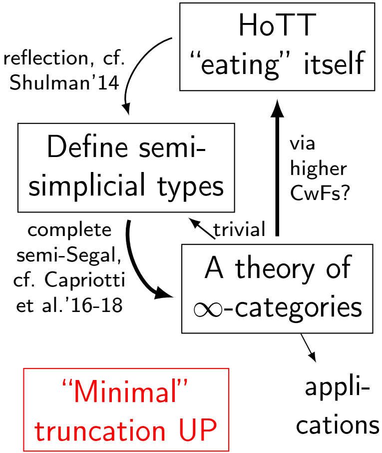
TT-IN-TT $\Rightarrow$ SST: generate SST terms, reflect them into the host type theory.

Open Problems



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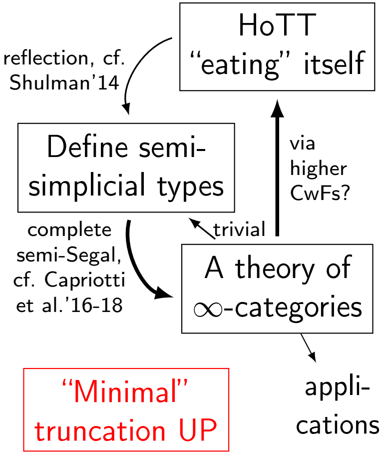


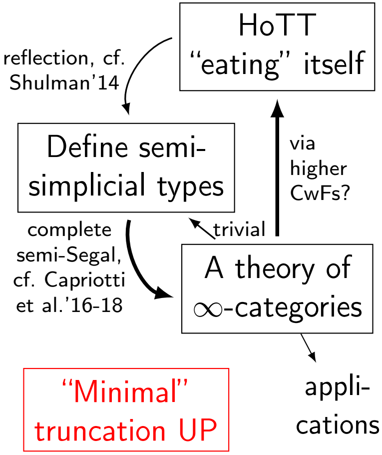
ELIMINATING TRUNCATIONS

What is $\|A\|_n \to B$
(for untruncated B)?

E.g. if B is set:

$$(\|A\|_{-1} \to B)$$
$$\simeq \; \Sigma(f : A \to B), \; \Pi(x, y : A), fx = fy$$

Open Problems



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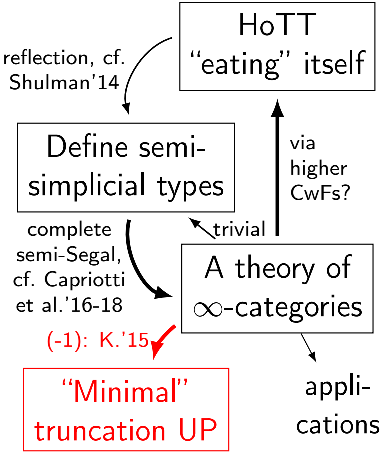
We have constructions of truncations (v Doorn, Rijke, K.) which give elimination principles into arbitrary types – but difficult to apply!

Open Problems



USING HIGHER CAT'S

- Write $\eta : \mathcal{U} \to \infty$CAT, i.e. $\eta(X)$ is the ∞-groupoid of X (RF replacement of X).
- Write coskel$_n$: ∞CAT $\to$ ∞CAT for the *[n]-coskeleton* (removes cells above dimension n)

CONJECTURE: $(\|A\|_n \to B) \simeq (\text{coskel}_{n+1}(\eta A) \to \eta B)$

RESULT:
Works for $n \equiv -1$ (K.'15)

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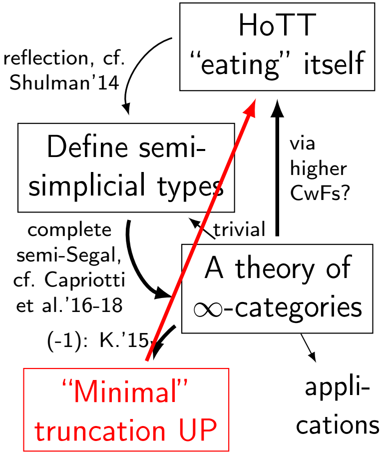

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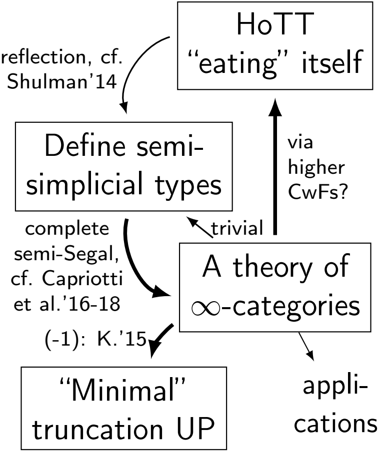

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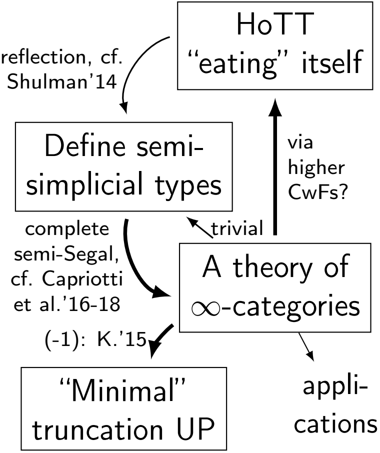

Open HoTT problems
(suspension of set, . . .)

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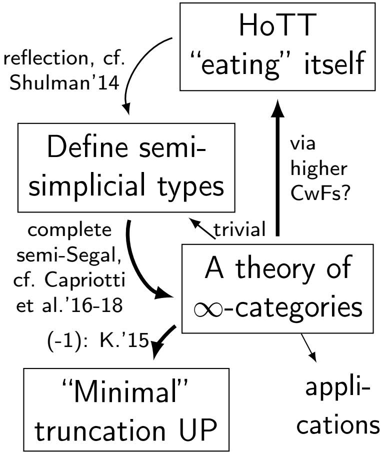


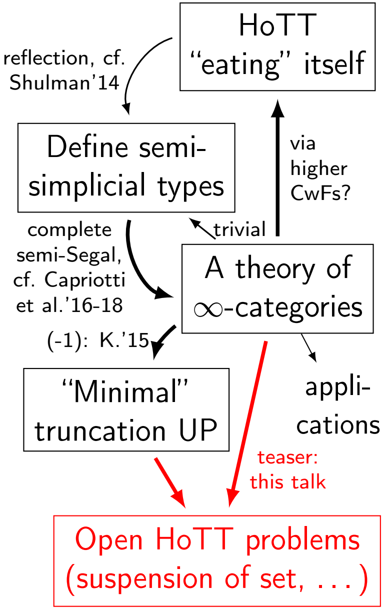
Open HoTT problems
(suspension of set, ...)

ELEMENTARY OPEN PROBLEMS

- Free ∞-group over a set: is it a set?
- Suspension of a set: is it a 1-type?
- Adding a path to a 1-type: what can we say?

What do these have to do with the other questions?

Open Problems



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"Elementary" Problems

Wedge of A-many circles:

$$A \rightrightarrows 1 \dashrightarrow WA$$

Or, as HIT:

data WA

base : WA

loops : $A \to \mathsf{base} = \mathsf{base}$

Free group $\mathsf{FG}(A)$ is $\Omega(WA)$.

If A is a set $\Rightarrow$ is $\mathsf{FG}(A)$ a set?

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Slightly more general:

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ \downarrow & & \vdots \\ 1 & \dashrightarrow & \Sigma A \end{array}$$

If A is a set, is ΣA a 1-type?

"Elementary" Problems (cont.)

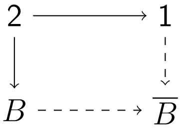
Adding a path:

$$\begin{array}{ccc} 2 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ B & \dashrightarrow & \overline{B} \end{array}$$

If B is a 1-type, is the pushout $\overline{B}$ still a 1-type?

"Elementary" Problems (cont.)

Adding a path:



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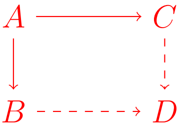
Generalization of the above:



If A is a set and B, C are 1-types, is D a 1-type?

"Elementary" Problems (cont.)

Adding a path:

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Generalization of the above:

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \dashrightarrow & D \end{array}$$

If A is a set and B, C are 1-types, is D a 1-type?

Notes: (1) Yes, if LEM; (2) Probably no, if we try to generalize further (thanks to P. Capriotti for the left example):

$$\begin{array}{ccc} 2 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ \left\|\mathbb{S}^2\right\|_2 + \left\|\mathbb{S}^2\right\|_2 & \dashrightarrow & \left\|\mathbb{S}^2\right\|_2 \vee \left\|\mathbb{S}^2\right\|_2 \end{array} \qquad \begin{array}{ccc} \mathbb{S}^1 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \dashrightarrow & \mathbb{S}^2 \end{array}$$

The Difficulty

How to characterize path spaces in
$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \vdots \\ B & \dashrightarrow & D \end{array}$$
?

If A is a set and B, C are 1-types, we want path spaces in D to be sets.

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(. . . so this does not answer the question.)

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The problem:

Forming "coherent quotients" without truncating is hard.

Quotienting by directed relations

What I need from a directed relation:

- a set $W : \mathcal{U}$, together with a function $\mathit{deg} : W \to \mathbb{N}$;
- a family $\leadsto : W \times W \to \mathcal{U}$;
- such that $(w \leadsto v) \to \mathit{deg}(w) > \mathit{deg}(v)$.
- for $(w, v, u : W)$ such that $w \leadsto v$ and $w \leadsto u$, we get $t : W$ together with $v \leadsto^{rt} t$ and $u \leadsto^{rt} t$.

Here, $\leadsto^{rt}$ is the refl-trans closure of $\leadsto$, as in:

$$
\begin{aligned}
&\text{data } \leadsto^{rt} : W \to W \to \mathcal{U} \\
&\qquad \text{nil} : \{w : W\} \to (w \leadsto^{rt} w) \\
&\qquad \text{cons} : \{w, v, u : W\} \to (w \leadsto^{rt} v) \\
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Quotienting by directed relations

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- for $(w, v, u : W)$ such that $w \rightsquigarrow v$ and $w \rightsquigarrow u$, we get $t : W$ together with $v \rightsquigarrow^{rt} t$ and $u \rightsquigarrow^{rt} t$.

Idea: We want $(W / \rightsquigarrow)$.

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$$\begin{aligned} \text{data } \rightsquigarrow^{rt} &: W \to W \to \mathcal{U} \\ \text{nil} &: \{w : W\} \to (w \rightsquigarrow^{rt} w) \\ \text{cons} &: \{w, v, u : W\} \to (w \rightsquigarrow^{rt} v) \\ &\to (v \rightsquigarrow u) \to (w \rightsquigarrow^{rt} u) \end{aligned}$$

General case – undirected relation

Consider any set X with $\sim: X \to X \to \mathcal{U}$.
Write $(X/\sim)$ for the set-quotient, with $[-] : X \to (X/\sim)$.

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Fact: For any $x, y : X$, the canonical map

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Fact: For any $x, y : X$, the canonical map

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Consequence: If Y is a 1-type, then the type

$$(X/\sim) \to Y \tag{2}$$

is equivalent to the type of triples (f, p, q) where

$$\begin{aligned} & f : X \to Y \\ & p : \Pi(a, b : X), (a \sim b) \to f(a) = f(b) \\ & q : \Pi(a : X), (l : a \sim^{rst} a), p^*(a, l) = \text{refl} \end{aligned} \tag{3}$$

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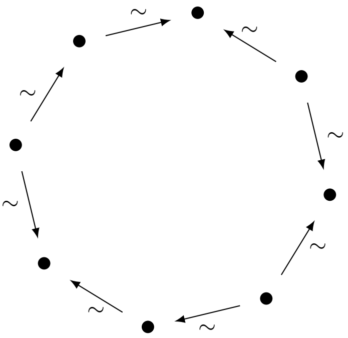
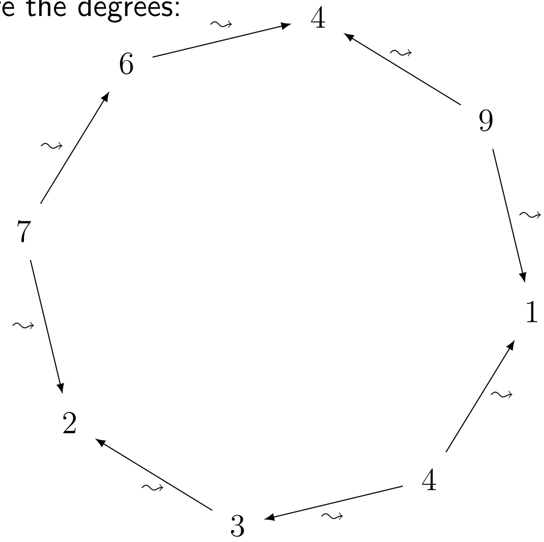
"Polygon" in X:



f: maps points to points in Y
p: maps lines to equalities in Y
q: polygon in Y is trivial

What do we gain from a directed relation?

"Polygon" in W, $\rightsquigarrow$ directed, vertices are the degrees:



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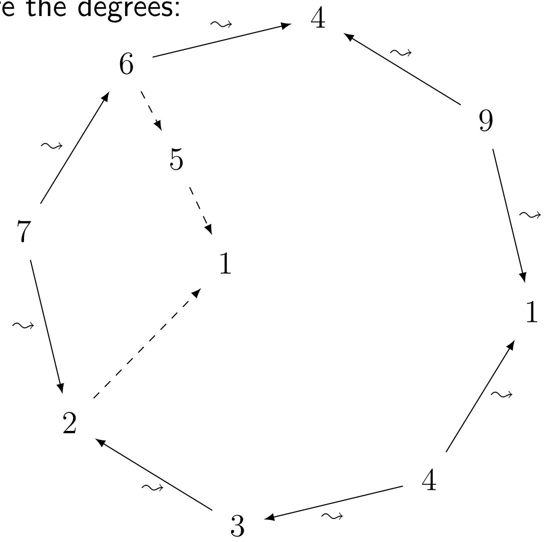

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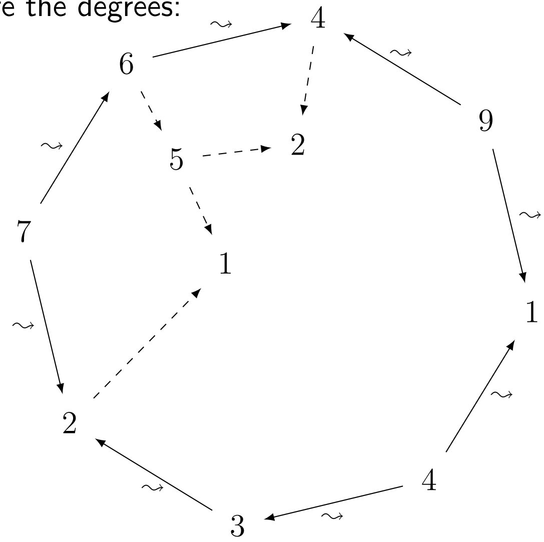
"Polygon" in W, $\rightsquigarrow$ directed, vertices are the degrees:



What do we gain from a directed relation (cont)?

Lexicographical order on ordered lists on $\mathbb{N}$ is well-founded

$\Rightarrow$ this process necessarily terminates with the trivial polygon

$\Rightarrow$ every "polygon" can be disassembled into "confluence polygons"

$\Rightarrow$ in

$$f : X \to Y$$
$$p : \Pi(a, b : X), (a \rightsquigarrow b) \to f(a) = f(b)$$
$$q : \Pi(a : X), (l : a \rightsquigarrow^{rst} a), p^*(a, l) = \text{refl}$$

it is enough if q quantifies over those shapes given from the confluence condition!

q : *confluence shapes* $\overset{p}{\mapsto}$ *commuting polygons*

What do we gain from a directed relation (cont)?

This gives a fairly easy way to construct functions

$$(W/\rightsquigarrow) \to Y$$

if Y is a 1-type!

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$$\begin{array}{ccc} A \text{ (set)} & \longrightarrow & C \text{ (1-type)} \\ \downarrow & & \vdots \\ B \text{ (1-type)} & \dashrightarrow & D \text{ (???)} \end{array}$$

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SvK theorem (Fav.-Shulm.): $(\text{lists}/\rightsquigarrow) \simeq \|\textit{eq-types in } D\|_0$

Now we can get: $(\text{lists}/\rightsquigarrow) \to \|\textit{eq-types in } D\|_1$

(which is the hard part of "2^{nd} hom-groups of D are trivial").

How to generalize?

$$f : X \to Y$$

$$p : \Pi(a, b : X), (a \sim b) \to f(a) = f(b)$$

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Next step (Y is a 2-type): q maps the boundary of any polyhedron whose sides are confluence shapes to a trivial polyhedron in Y?
Seems difficult. . .

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Thank you for your attention!

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