Univalent Higher Categories via Complete Semi-Segal Types

Paolo Capriotti Nicolai Kraus

POPL'18, Los Angeles, Wed 10 Jan 2018

Setting

Martin-Löf style type theory

formal systems of terms and dependent types,

e.g.: $\Pi(n : \mathbb{N}).\Sigma(p, q : \text{Primes}).(p + q = n + n + 4)$

What is it good for?

Programming — Proof assistants — Foundation

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Agda

`wiki.portal.chalmers.se/agda`

LƎ∀N

`leanprover.github.io`

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What do we want to implement?

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Defining categories, first attempt

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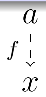

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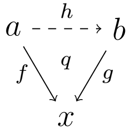

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  - associativity: needs α and ???

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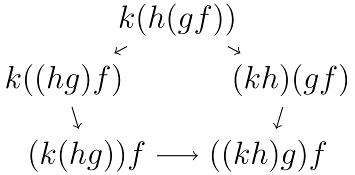

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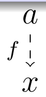

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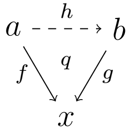

  - composition: needs $\circ$ and α
  - associativity: needs α and **MacLane's pentagon** (familiar from bicategories)

How to solve the

$$k(h(gf))$$

$$\swarrow \qquad \searrow$$

$$k((hg)f) \qquad\qquad (kh)(gf)$$

$$\downarrow \qquad\qquad \downarrow$$

$$(k(hg))f \longrightarrow ((kh)g)f$$

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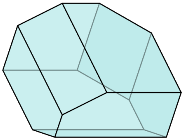
Naïve solution: Add the pentagon to the definition of a category.

Problem: Now we **can** derive associativity for $\mathcal{C}/x$, but we **cannot** derive the pentagon for $\mathcal{C}/x$. For this, we would need the associahedron K_5.



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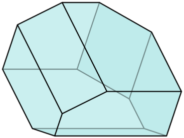

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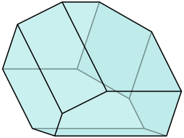

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Side remark: Ahrens-Kapulkin-Shulman (2015): categories where morphisms satisfy UIP. These are well-behaved, but important examples are not captured.

Contributions

A definition for higher categories in type theory: *Complete semi-Segal types*. More precisely: $(n,1)$-categories, $n \leq 2$ done explicitly, n externally fixed, $(\infty,1)$ possible in some extensions of "standard HoTT".

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- **Semisimplicial types** (HoTT 2012) give raw data
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For the special case of $n \leq 2$, we show that our definition is equivalent to the "manual" definition (e.g. Ahrens-Kapulkin-Shulman 2015), in Agda.

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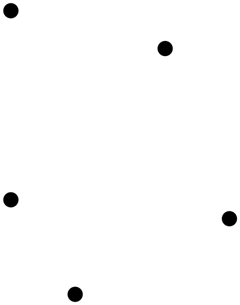

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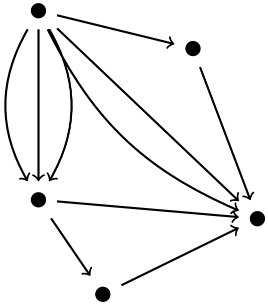

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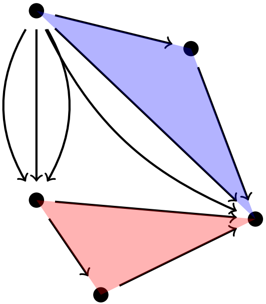
- For any "empty triangle" a type of *fillers*,

$$A_2 : \Pi(a, b, c : A_0).A_1(b, c) \to A_1(a, b) \to A_1(a, c) \to \text{Type}.$$



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in general:

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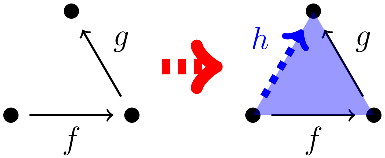
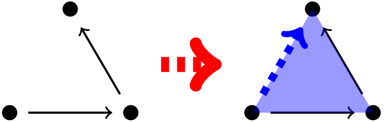
h_2 says: even horn (f, g) has unique filler:



Segal condition

Def.: A semisimplicial type $(A_0, \ldots, A_n)$ fulfils the ***Segal condition*** if, for every $2 \leq k \leq n$, every Λ_1^k-horn has a unique filler.

$k = 2$:



(composition!)

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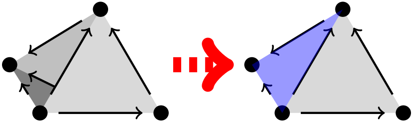

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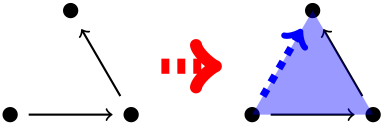

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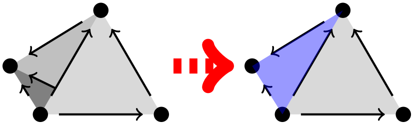

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$k = 4$ (cannot draw it) gives the pentagon equality.

Disclaimer: Of course, the connection between horn fillers and composition structure is known (Joyal/Rezk/Lurie/…), we merely checked that this works in type theory.

Completeness

Have: composition structure.
Now: identities.
Strategy: Lurie (2009) and Harpaz (2015).

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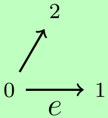
Def: $e : A_1(x, y)$ is ***neutral*** if all horns of the form



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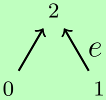
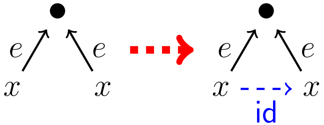


have unique fillers.

A semisimplicial type is ***complete*** if every point has exactly one outgoing neutral edge.

We get identities on x like this:



Conclusions

Def: A ***complete semi-Segal*** n***-type*** is a semisimplicial type $(A_0, \ldots, A_{n+2})$ that satisfies:

$$\left.\begin{array}{l} \bullet \text{ Segal condition} \\ \bullet \text{ completeness} \\ \bullet \text{ truncation (highest level trivial)} \end{array}\right\} \text{(propositions)}$$

A definition on its own is not very useful. Potential applications of higher categories (all wip):

- formalized higher categorical model of type theory (∞-CwF)
- constructing higher inductive types
- …?

Thank you for your attention!