Equality in the Dependently Typed Lambda Calculus: An Introduction to Homotopy Type Theory

or: Connecting Topology and Logic with Category Theory

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Typed $\lambda$ Calculus

**Natural Deduction**

$$\frac{A \to B \quad A}{B}$$

$$\frac{B}{A \to B}$$

**Curry-Howard**

$$\cong$$

**Type Theory**

$$\frac{\Gamma \vdash f : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash f \ u : B}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x. t : A \to B}$$
Dependently Typed $\lambda$ Calculus

Types may depend on terms:

$$\text{Vec } A \ n$$

are Lists over $A$ with length $n$. 
Dependently Typed $\lambda$ Calculus

Natural Deduction \[\cong\] Type Theory \[\text{special case}\]

\[\exists_{x \in A} B\] \[\Sigma(x:A).B\] \[A \times B\]

\[\forall_{x \in A} B\] \[\Pi(x:A).B\] \[A \rightarrow B\]

Usage, e.g. Agda & Epigram: proof assistants, formal verification, proof-carrying code
Problems...

- Typechecking requires Computation.
- Equality is no longer decidable in general.
- We want decidable typechecking.
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Two kinds of Equality!

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Definitional Equality
“Real” decidable equality such as \((\lambda a.b)x =_\beta b[x/a]\)

Propositional Equality
Equality needing a proof
...and Answers

Two kinds of Equality!

Definitional Equality
“Real” decidable equality such as $(\lambda a.b)x \beta b[x/a]$

Propositional Equality
Equality needing a proof
Propositional Equality

\[
\Gamma \vdash x, y : A \\
\Gamma \vdash \text{Id}_A x y : \text{type} \quad \text{Form} \\
\Gamma \vdash x : A \\
\Gamma \vdash \text{refl}_x : \text{Id}_A x x \quad \text{Intro}
\]
Propositional Equality

\[ \Gamma \vdash A : \text{type} \]
\[ \Gamma, x, y : A, p : \text{Id}_A x y \vdash M(x,y,p) : \text{type} \]
\[ \Gamma, r : A \vdash m : M(r, r, \text{refl}_r) \]
\[ \Gamma \vdash a, b : A \]
\[ \Gamma \vdash q : \text{Id}_A a b \]
\[ \Gamma \vdash J M m a b q : M(a,b,q) \]

\[ \text{Elim } (J) \]

\[ J M m a a \text{ refl}_a = m a \]

\[ \text{Comp} \]
Subst from J

- $P : A \rightarrow \text{Set}$ and $a, b : A$.
- $q : \text{Id}_A a b$
- $p : P a$
- Can we get something of type $P b$?

I.e. is $(P : A \rightarrow \text{Set}) \rightarrow (a, b : A) \rightarrow \text{Id}_A a b \rightarrow P a \rightarrow P b$ inhabited?

Sure! Using $J$ with

$$M = \lambda x y p . P x \rightarrow P y$$

$$m = \lambda x . x$$

Call it $\text{subst}$. 
Subst from J

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(9/24) COMPUTING 2011 – 2011-10-21
How many inhabitants can $\text{Id}_A a b$ have in general?

For some time, it was assumed that there is at most one (UIP), i.e. given $p, q : \text{Id}_A a b$, the type $\text{Id} p q$ is inhabited.

Hofmann-Streicher groupoid model: not derivable from $J$. 
How many inhabitants can $ld_A \ a \ b$ have in general?

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Uniqueness of Identity Proofs - Refuted

\[ a, b, d, e : A \quad c, f : B \]
\[ s : \text{Id}_A a a \quad p, q : \text{Id}_A a b \quad u, v : \text{Id}_A b e \quad \ldots \]
UIP is weird anyway

\[ BOOL = \{ \text{true, false} \} \]

isomorphisms:

\[ id : BOOL \rightarrow BOOL \]

\[ \neg : BOOL \rightarrow BOOL \]

So, identity equals negation?!
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isomorphisms:

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\[ \neg : \text{BOOL} \to \text{BOOL} \]

So, identity equals negation?!
Extensionality

Given:

- \( f : A \rightarrow B \)
- \( g : A \rightarrow B \)
- \( p : \Pi(x : A).Id_B (fx) (gx) \)

Can we construct something of type \( Id_{A\rightarrow B} f g \) (Leibniz)? No!
Given:

- \( f : A \rightarrow B \)
- \( g : A \rightarrow B \)
- \( p : \prod(x : A).Id_B(fx)(gx) \)

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Idea: Adding extensionality as additional axiom.

But then, assume \( p \) is a (nontrivial) equality proof using this axiom.

Consequence:

\[
\text{subst} \ (\lambda h \to \mathbb{N}) \ p \ 0
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Non-canonical natural numbers!
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Non-canonical natural numbers!
Univalence and weak omega groupoids

Vladimir Voevodsky
Voevodsky’s suggestion

Do not use \textit{UIP}
...because it is weird and has undesirable consequences!

Do not use the Extensionality Axiom!
...because of the same reason!

Use Univalence instead!
...because it is better - as we will see in a moment!
Voevodsky’s suggestion

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Lumsdaine’s and v.d.Berg’s result

for example:

- $a := b := x$
- $p := p' := \text{refl}_x$
- $H := H' := \text{refl}_{\text{refl}_x}$
- $\text{refl}_{\text{refl}_{\text{refl}_x}}$
- $\ldots$
Lumsdaine’s and v.d.Berg’s result

Univalence and weak omega groupoids

Weak $\omega$ groupoid

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Weak $\omega$ groupoid
A very well-known structure...  

...in Topology!

(source: Wikipedia)
A disc

a (nondependent!) type - we call it $X$

a topological space - we call it $X$
A disc

two terms

two points
A disc

? a path
A disc

\[ a, b : X \]
\[ p : \text{Id} \; a \; b \]

\[ a, b \in X \]
\[ p : [0, 1] \rightarrow X \]
\[ p(0) = a \]
\[ p(1) = b \]
A disc

\[ p^{-1} : \text{Id} \ b \ a \]

\[ p^{-1} : [0, 1] \to X \]

\[ p^{-1}(t) = p(1 - t) \]
A disc

\[ p : a \equiv b \]
\[ q : \text{Id } b \rightarrow c \]

\[ a, b \in X \]
\[ p : [0, 1] \rightarrow X \]
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\[ q : [0, 1] \rightarrow X \]
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\[ q(1) = c \]
A disc

\[ q \circ p : \text{Id} a c \]

\[ q \circ p : [0, 1] \to X \]

\[ x \mapsto \begin{cases} 
  p(2x), & x < 0.5 \\
  q(2x - 1), & \text{else} 
\end{cases} \]
Another set

\(Id\ a\ c\ \text{not inhabited}\)

Not path-connected
A disc

\[ p, p' : \text{Id} \ a \ b \]

\[ p, p' : [0, 1] \to X \]
A disc

\[ H : [0, 1]^2 \to X \]
\[ H(0, \cdot) = p \]
\[ H(1, \cdot) = p' \]
\[ H(t, 0) = a \]
\[ H(t, 1) = b \]
A disc

\[ H : [0, 1]^2 \rightarrow X \]
\[ H(0, \cdot) = p \]
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\[ p : [0, 1]^1 \rightarrow X \]
\[ a : [0, 1]^0 \rightarrow X \]
A ring

\[ H : \text{Id } p \ x \ p' \]

\[ H : [0, 1]^2 \rightarrow X \]
\[ H(0, \cdot) = p \]
\[ H(1, \cdot) = p' \]
\[ H(t, 0) = a \]
\[ H(t, 1) = b \]
A disc

\[ H : \text{Id} \, p \, p' \]

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A disc

\[ H' : \text{Id}_p \, p \, p' \]

\[ H'(0, \cdot) = p \]
\[ H'(1, \cdot) = p' \]
\[ H'(t, 0) = a \]
\[ H'(t, 1) = b \]
A disc

\[ K : \text{Id } H' \Rightarrow H \]

\[ K : [0, 1]^3 \rightarrow X \]

\[ K(0, \cdot, \cdot) = H' \]

\[ \ldots \]
Putting it together

\[ \begin{array}{ccc}
  a & \xrightarrow{p} & b \\
  H' & \xrightarrow{K} & H \\
  \end{array} \]

\[ \begin{array}{ccc}
  p & \xrightarrow{K} & p' \\
 \end{array} \]
The (canonical) mapping from equalities to weak equivalences is a weak equivalence. 

- No need for $\text{UIP}$
- Extensionality
- Only canonical members of $\mathbb{N}$
- A "completely natural axiom" so that everything works as in homotopical intuition
Voevodsky again

Univalence Axiom

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Summary

Hopes:

Homotopic Models:
- new results and intuition in both type and homotopy theory
- better understanding of the connection between logic and topology

Univalence:
- avoiding a couple of problems in a natural way
  - UIP
  - Extensionality
  - Canonicity of natural numbers
- better foundation than Set Theory for (constructive) mathematics
- at the same time, natively supported by proof assistants
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(Other) People I want to mention

- Thorsten Altenkirch
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- Nicola Gambino
- Richard Garner
- Chris Kapulkin
- Dan Licata
- Mike Shulman
- Thomas Streicher
- Michael Warren
- ... and many more
Even more people I want to Thank

You.