Functions out of Higher Truncations

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on joint work with

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In Homotopy Type Theory
(i.e. dependent type theory with \(\Sigma, \Pi, =,\) univalence, HITs), how can we
Construct functions \(\|A\|_n \to B\)
if \(B\) is \((n + 1)\)-truncated?
In Homotopy Type Theory

✓ ✓ (clear, I hope) (does not matter)

(i.e. dependent type theory with \(\Sigma, \Pi, =\), univalence, HITs), how can we

Construct functions \(\|A\|_n \to B\)

? ? (will explain)

if \(B\) is \((n + 1)\)-truncated?
Reminder: Identity/Equality Types in Martin-Löf’s Dependent Type Theory

- If $A$ is a type and $x, y : A$, then $x = y$ is also a type (a.k.a. $x =_A y$ or $\text{Id}_A(x, y)$)
- Does UIP hold? I.e. if $p, q : x = y$, do we automatically get $p = q$?
- Hofmann-Streicher 1994: No! [LICS Test of Time Award 2014]
- Types can have non-trivial higher structure (fist step of birth of Homotopy Type Theory)
Introduction: Truncation Levels and Truncations

- “being $n$-truncated” is a property of types [due to VV]
- intuition: “trivial on levels $\geq n$”
- Def:
  1. $A$ is ($-2$)-truncated iff $A \simeq \text{Unit}$
  2. $A$ is ($n + 1$)-truncated iff $x = y$ is $n$-truncated
     $\forall x, y : A$
- basic lemma: $A$ is $n$-truncated $\Rightarrow A$ is $(n + 1)$-truncated
- examples:
  - ($-2$)-truncated a.k.a. contractible: $\text{Unit}$
  - ($-1$)-truncated a.k.a. propositional: $\emptyset$
  - 0-truncated a.k.a. set, satisfying UIP: $\mathbb{N}$, $\text{Bool}$, …
  - 1-truncated: universe of sets
- …
Introduction: Truncation Levels and Truncations

- $\|A\|_n \to B \simeq (A \to B)$ if $B$ is $n$-truncated
- Intuition: $\|\_\|_n$ "truncates" a type (thus "loses information"!), have $\|\_\| : A \to \|A\|_n$
Back to the topic of this talk

So, how to get a map \( \|A\|_n \to B \) in general?
Or: When does \( f : A \to B \) factor through \( \|A\|_n \)?

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \\
\|A\|_n & \xrightarrow{f'} & B
\end{array}
\]

* always, if \( B \) is \( n \)-truncated
* this paper:

**Theorem**

If \( B \) is \((n+1)\)-truncated:

\[
f : A \to B \text{ factors through } \|A\|_n
\]

iff

\( f \) induces trivial maps on all \((n+1)\)-st loop spaces.
Special cases of the result

**Special case \( n = -1 \)**

If \( B \) is 0-truncated, i.e. has unique identity proofs:

\[
f : A \to B \text{ factors through } \|A\|_{-1}
\]

iff

\[
f \text{ is weakly constant: } \prod_{x,y:A} f(x) = f(y).
\]

\[
A \xrightarrow{f} B
\]

\[
\|A\|_{-1} \xrightarrow{|-|} f' \xrightarrow{f} B
\]

\( f \) must be weakly constant.

This was already known

[K-Escardó-Coquand-Altenkirch 2014].
Special cases of the result

Special case $n = 0$

If $B$ is 1-truncated:

$$f : A \to B$$

factors through $\|A\|_0$

iff

$$ap_f : (x = y) \to (f(x) = f(y))$$

is weakly constant.

Known from the Rezk completion

[Ahrens-Kapulkin-Shulman 2014].
Special cases of the result

Special case \( n = 1 \)

If \( B \) is 2-truncated:

\[ f : A \rightarrow B \] factors through \( \|A\|_1 \)

iff

\( f \) introduces trivial maps on all second homotopy groups / loop spaces:

\[ \text{ap}_f^2 : \Omega^2(A, a) \rightarrow \Omega^2(B, f(a)) \] weakly constant.

This (and all other cases) are new.
Two proofs of our result:

* first proof:
  - given $f' : \|A\|_n \to B$, we do get $f : A \to B$ which is trivial on all $(n+1)$-st loop spaces
  - if $A$ is $n$-connected, this map is an equivalence
  - piece together maps on the “$n$-connected components” of $A$

* second proof:
  - construct a higher inductive type $H^{A,n}$
  - show that $H^{A,n}$ has a suitable elimination principle
  - show $H^{A,n} \simeq \|A\|_n$
Overview: Open and Solved Cases

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Solved in: [K-Escardó-Coquand-Altenkirch 2014], [Ahrens-Kapulkin-Shulman 2014], [K 2015], [HERE]