Omega Constancy and Truncations

Nicolai Kraus

Functional Programming Lab, University of Nottingham

04/07/14

Higher Homotopical Structure of a Type

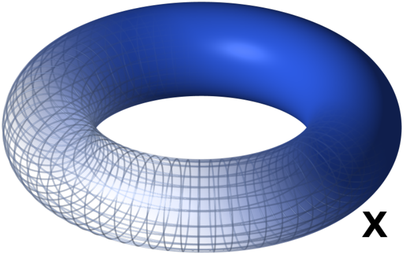
Consider a **type** X

...and two **points** $a, b : X$

$\Rightarrow$ we get a new **type** $a = b$.

Elements are **paths**, e.g.
$p : a = b$ or $q : a = b$

$\Rightarrow$ we get a new **type**
$p = q$, the inhabitants of which are **2-paths**.



Definition: if n such iterations always lead to a type that is isomorphic to the unit type, we say that X is an $(n-2)$-*type* or $(n-2)$-*truncated*.

The Truncation Monad

- (-1)-types are called *propositions*. They have the property that all their inhabitants are equal.
- A type theory can have a monad $\|-\|$ which turns any type into a proposition (which says that this type is inhabited).
- Concretely:
  - $\|A\|$ is propositional
  - $A \to \|A\|$
  - If B is propositional, then

$$(A \to B) \to (\|A\| \to B).$$

(That implies $\|A\| \to (A \to \|B\|) \to \|B\|$.)

Eliminating out of Truncations

Goal: a function $\|\mathbf{A}\| \to \mathbf{B}$. How do we get it?

If B is a proposition, then $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is enough. Interpretation:

"$f(a)$ does not depend on the concrete choice of $a : A$ because B does not have distinguishable elements anyway."

This suggests that f should be constant (if we want to drop the condition on B).

Constancy

But what is **"constant"**?

First try:

$$\mathsf{const}_f \;:\equiv\; \forall(x, y : A).\, f(x) = f(y)$$

Indeed, we can prove:

Theorem

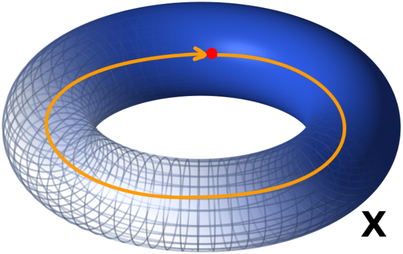
$$\left(\Sigma_{f:A\to B}\, \mathsf{const}_f\right) \;\simeq\; (\|A\| \to B)$$

if B is a 0-type.

Constancy

But look at this:

A function $1 \to$ Torus with a proof const_f:



We have to ask for more than that!

Coherence Conditions

The paths have to “fit together”.
Given $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ and $\mathbf{c} : \mathsf{const}_\mathsf{f}$, define

$$\mathsf{coh}_{f,c} \;:\equiv\; \forall (x, y, z : A).\, c(x, y) \cdot c(y, z) = c(x, z)$$

We can prove:

Theorem

$$\left(\Sigma_{f:A\to B}\, \Sigma_{c:\mathsf{const}_f}\, \mathsf{coh}_{f,c}\right) \;\simeq\; \left(\|A\| \to B\right)$$

if B is a 1-type.

The General Case

- If we do not know anything about B, we need an "infinite Σ-type".
- This can be done in a theory with certain Reedy-limits.
- Corollary: If B is an n-type, the first n conditions are sufficient.
- This case can be formalised in a proof assistant, for any n.
- Main contribution:

  a generalised universal property of the prop. truncation.
- P. Capriotti and I try to do the same for higher truncation, which requires a very different approach.

Questions?

Thank you!