Truncation Levels in Homotopy Type Theory
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My time as a PhD student

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Thank you!

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Type theory

Formal systems for programming, proving, formalising, foundation of mathematics

Central: $\mathbf{x} : \mathbf{A}$, “$\mathbf{x}$ is a term of type $\mathbf{A}$” (in some context)

→ interpretations:
☆ $\mathbf{A}$ is a set and $\mathbf{x}$ an element [Russel, 1903]
☆ $\mathbf{A}$ is a problem and $\mathbf{x}$ a solution [Brouwer–Heyting, Kolmogorov, 1932]
☆ $\mathbf{A}$ is a proposition and $\mathbf{x}$ a proof [Curry-Howard, 1969]
☆ $\mathbf{A}$ is a space and $\mathbf{x}$ a point (in case of MLTT)
  early form: Hofmann–Streicher 1996;
  Voevodsky (from 2006/10);

⇒ Homotopy type theory / Univalent foundations

I have adapted this list from Pelayo-Warren, “Homotopy type theory and Voevodsky’s univalent foundations”.
Truncation levels (Voevodsky: h-levels)

* Truncation levels express (an upper bound of) the homotopical complexity of types, starting as follows:

- **level -2**: “contractible”, equivalent to Unit
- **level -1**: “propositional”, contractible equality types
- **level 0**: a “set”, propositional equality types
- **level 1**: a “groupoid”, equality types are sets
Non-truncated types (1)

Well-known fact: In Martin-Löf type theory with the univalence axiom, the lowest universe $\mathcal{U}_0$ is not a set.

Proof: $\text{Bool}$ is equivalent to itself in two different ways (identity and negation), thus univalence gives two different elements of $\text{Bool} = \text{Bool}$.

Open problem of the special year in Princeton (2012):
Given a hierarchy $\mathcal{U}_0 : \mathcal{U}_1 : \mathcal{U}_2 : \ldots$ of univalent universes, can we construct types that are provably not $n$-truncated?

This is indeed the case (I presented a proof in Princeton in April 2013).
Extended answer (K. and Sattler 2013/2015):

* The universe $\mathcal{U}_n$ is not $n$-truncated.
* $\mathcal{U}_n$, restricted to $n$-truncated types, is a “strict” $(n + 1)$-type.
* With some additional effort, we get a strict $n$-type which has trivial homotopy groups on all levels except $n$.
* Note: It is consistent to assume that $\mathcal{U}_n$ is $(n + 1)$-truncated, i.e. the first two results are optimal. The third “wastes” one universe level.
**U\_1** is not 1-truncated, proof

(0) Assume **U\_1** is 1-truncated.
(1) Set \(L \equiv \Sigma(X : \mathcal{U}_0).(X = X)\).
(2) If \(\mathcal{U}_1\) is 1-truncated, then \(L = L\) is a set.
(3) Then, \(\text{refl}_L = \text{refl}_L\) is a proposition.
(4) Univalence-translated: \((\text{id}_L, \text{e}_{\text{id}}) = (\text{id}_L, \text{e}_{\text{id}})\) is a proposition.
(5) Simplifies to: \(\text{id}_L = \text{id}_L\) is a proposition.
(6) By function extensionality: \(\Pi_{x : L}(x = x)\) is a proposition.
(7) By unfolding and currying: \(\Pi_{A : \mathcal{U}_0} \Pi_{p : A = A}(A, p) = (A, p)\) is a proposition.
(8) Rewrite with standard lemmas:
\[\Pi_{A : \mathcal{U}_0} \Pi_{p : A = A} \Sigma(q : A = A).(p \cdot q = q \cdot p)\] is a proposition.
(9) ... but this type has multiple elements, e.g.
\(\lambda A.\lambda p.(\text{refl}_A, \_ )\) and \(\lambda A.\lambda p.(p, \_ )\).
$\mathcal{U}_n$ is not $n$-truncated, some ideas

Recall: $\mathcal{U}_n$ is $n$-truncated $\iff \Omega^{n+1}(\mathcal{U}_n, X)$ is contractible.

- By induction on $n$.
- Consider $(n + 1)$-loops in $\mathcal{U}_n^n$, i.e.:
  \[ \Sigma(A : \mathcal{U}_n^n).\Omega^{n+1}(\mathcal{U}_n^n, A). \]
  Here, $\mathcal{U}_n^n$ is $\mathcal{U}_n$ restricted to $n$-truncated types (crucial trick!).
- We can “move between universes” with our local-global looping principle:
  \[ \Omega^{n+2}(\mathcal{U}, A) \simeq \Pi_{a : A} \Omega^{n+1}(A, a) \]
  (this is simple, essentially function extensionality).
New topic: Propositional truncation

* In HoTT: we consider an operation $\| - \|$ which turns a type into a propositional type. Roughly: reflector of the subcategory of propositional types.

* We only know how to construct a function $\|A\| \to B$ if $B$ is propositional.

* The (in my opinion) main result of my thesis is:

$$ (\|A\| \to B) \simeq \mathcal{U}^{\Delta^+}_{\mathcal{C}}(\mathcal{T}A, \mathcal{E}B) $$

where $\mathcal{E}B$ is the Reedy fibrant replacement of $(\text{const}) B$ and $\mathcal{T}A$ the $[0]$-coskeleton of $A$.

Very much related to 6.2.3.4 in Lurie’s *Higher Topos Theory* and 7.8 in Rezk’s *Toposes and Homotopy Toposes*.

* I will not talk about this today. Instead, I conclude with a fun result.
A “mysterious puzzle”

Consider the function $|\_| : \mathbb{N} \to \|\mathbb{N}\|$. There is a term myst such that $\prod_{n:\mathbb{N}} \text{myst}(|n|) = n$.

- Consequence: $0 = \text{myst}(|0|) \neq \text{myst}(|1|) = 1$ How?

- Solution: the type of myst is **not** just $\|\mathbb{N}\| \to \mathbb{N}$. In fact, $\text{myst} : \prod_{x:\|\mathbb{N}\|} C(x)$ with a **very** complicated $C$. It just happens that $C(|n|) \equiv \mathbb{N}$!

- Here’s how to do it:
  Observe that $(\mathbb{N}, 0) = (\mathbb{N}, n)$ as pointed types. Define $f : \mathbb{N} \to \Sigma(Y : U.)( (\mathbb{N}, 0) = Y )$
  
  $n \mapsto ((\mathbb{N}, n), \_)$

- $f' : \|\mathbb{N}\| \to \Sigma(Y : U.)( (\mathbb{N}, 0) = Y )$

- define $\text{myst} :\equiv \text{snd} \circ \text{fst} \circ f'$. 
Conclusions

I have done some stuff about truncation levels in type theory, and I really enjoyed my time as a PhD student.

Thank you!