On Kock's (and Sattler's) fat Delta

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(based on joint work with Tom de Jong and Stiéphen Pradal)

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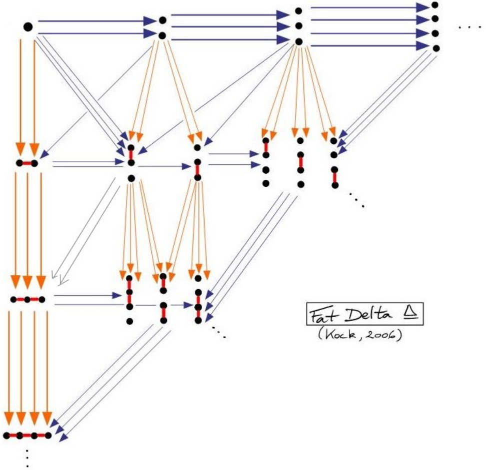
simplex category Δ
(positive finite ordinals)



Joachim Kock's[a] *fat Delta* $\underline{\Delta}$, a direct replacement[b] of Δ

A variation ("Sattler's fat Delta") was introduced with a different motivation.[c]

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[a] Kock, *Weak identity arrows in higher categories*, 2006.

[b] Sattler, *Kock's fat Δ is a direct replacement of Δ*, 2017.

[c] Kraus and Sattler, *Space-valued diagrams, type-theoretically*, 2017.

Kock's motivation for fat Delta

Joachim Kock, *Weak identity arrows in higher categories*, 2006.

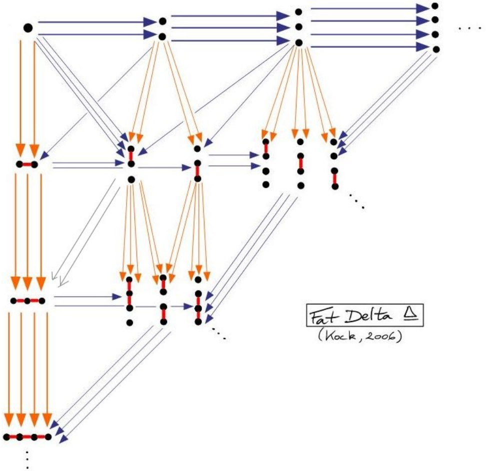
1. 1-categories: strict units, strict associativity
$$f \circ \mathrm{id} = f = \mathrm{id} \circ f \quad \text{and} \quad (h \circ g) \circ f = h \circ (g \circ f)$$
2. higher categories: strict/weak units, strict/weak composition structure
3. Kock's weak higher categories (based on $\underline{\Delta}$): weak units, strict associativity
$$f \circ \mathrm{id} \simeq f \simeq \mathrm{id} \circ f \quad \text{and} \quad (h \circ g) \circ f = h \circ (g \circ f)$$

Simpson's conjecture: (weak units, weak composition) $\simeq$
(weak units, strict composition)

Carlos Simpson, *Homotopy types of strict 3-groupoids*, 1998.

Kock's motivation for fat Delta (II)

Model for "weak units, weak associativity":
Presheaves with horn lifting condition



Model for
"weak units, strict associativity":
Presheaves that send orange maps to equivalences, with strict Segal condition

The type theory motivation for fat Delta

Type theorists like *indexed* presentations (rather than *fibered*):

☹ A graph is a set V, a set E, and maps $s, t : E \to V$.

☺ A graph is a set V and a family $E : V \to V \to \mathrm{Set}$.

Reason: Avoids equalities.

$$\begin{tikzcd}
\bullet\, V \\
\bullet\, E \arrow[u, bend left] \arrow[u, bend right]
\end{tikzcd}$$

Type theorists consider types, and types $\sim$ spaces.[1] Everything up to homotopy $\Rightarrow$ there's (in general) no type of 1-functors from a given category $\mathcal{C}$ to the category of types.

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[1] Your mileage may vary.

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$\Rightarrow$ there's (in general) no type of 1-functors from a given category $\mathcal{C}$ to the category of types.

General approach: interpret arrows as type dependencies. Another example is Δ_+^{op} ("semisimplicial types") restricted to objects [0], [1], [2]:

$$\begin{tikzcd} {[0]}\bullet \\ {[1]}\bullet \arrow[u, shift left] \arrow[u, shift right] \\ {[2]}\bullet \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] \end{tikzcd}$$

$A_0 : \mathrm{Type}$

$A_1 : A_0 \times A_0 \to \mathrm{Type}$

$A_2 : (x, y, z : A_0) \to A_1(x, y) \to A_1(y, z) \to A_1(x, z) \to \mathrm{Type}$

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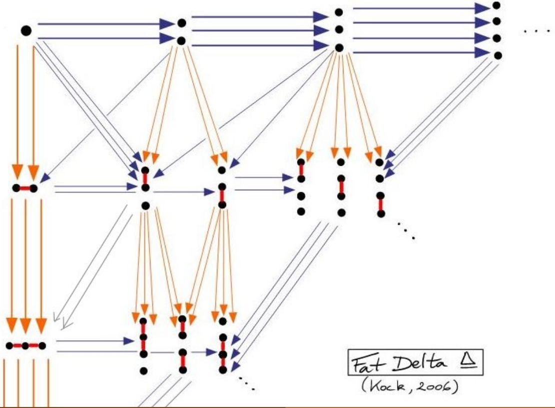
[1] Your mileage may vary.

The type theory motivation for fat Delta (II)

- This representation only works for inverse categories
  (maps only "go downwards," i.e. no cyclic dependencies).
- It only captures Reedy fibrant diagrams, but we work "up to homotopy" anyway.

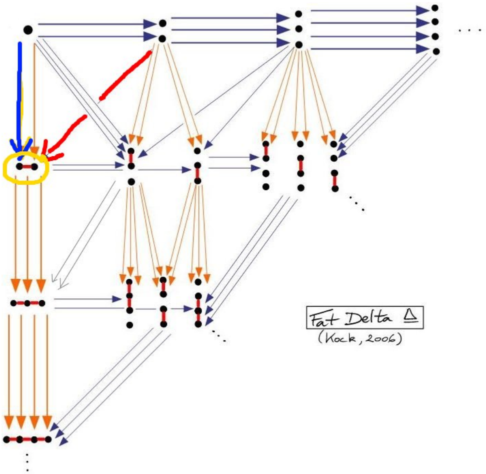
"cyclic dependencies"



no "cyclic dependencies"

(It's a direct category, i.e.
its opposite is inverse)

The type theory motivation for fat Delta (III)



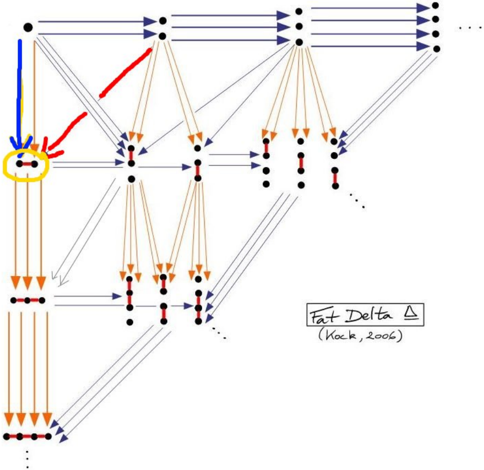
Extend prev. example from Δ_+^{op} to $\underline{\Delta}^{\mathrm{op}}$:

$A_{\bullet - \bullet}(x, y, p)$ means "p is equivalence";

blue arrow being an equivalence means "exactly one outgoing equivalence";

and so on.

The type theory motivation for fat Delta (III)



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forthcoming PhD thesis of Stiéphen Pradal:

model structure on $[\underline{\Delta}^{\mathrm{op}}, \mathrm{Spaces}]$

$\simeq$ (Quillen equivalent)

Harpaz's model structure

(and thus also to Rezk's model structure on $[\Delta^{\mathrm{op}}, \mathrm{Spaces}]$)

Fat delta

Definition (Kock's fat Delta)

The category $\underline{\Delta}$ is defined as $(\Delta_{-}^{\rightarrow})_{\mathrm{Mono}}$, where Δ_{-} is the restriction of the simplex category to its epimorphisms. Thus

objects: epimorphisms $\eta : [m] \twoheadrightarrow [n]$ in Δ

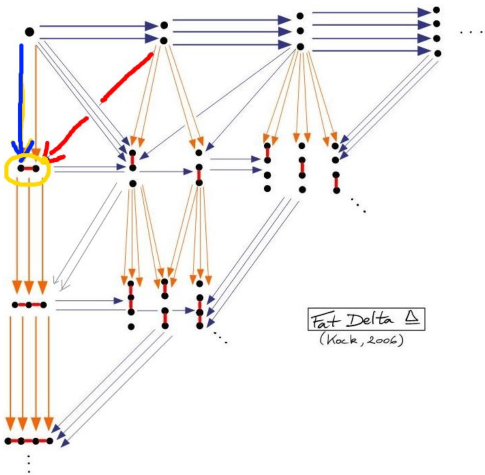
morphisms: $f : \eta_0 \to \eta_1$ is a commutative squares in Δ

$$\begin{tikzcd}
{[m_0]} \arrow[r, hook, "\overline{f}"] \arrow[d, two heads, "\eta_0"'] & {[m_1]} \arrow[d, two heads, "\eta_1"] \\
{[n_0]} \arrow[r, "\underline{f}"'] & {[n_1]}
\end{tikzcd}$$

whose horizontal top arrow is a monomorphism.

Sattler's fat Delta: The same, but the top arrow merely *exists*.

Kock's vs Sattler's fat Delta



Kock's fat Delta:

$A_{\bullet-\bullet}(x, y, p)$ means "p is equivalence";

blue arrow being an equivalence means "exactly one outgoing equivalence";

and so on.

Sattler's fat Delta:

identify vertical arrows that are "equal up to connected components"

$A_{\bullet-\bullet}(x, p)$ means "p is an identity";

blue arrow being an equivalence means "one identity per object";

and so on.

Relative semi-ordinals

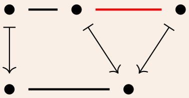
- A semi-ordinal is a semi-category associated with a total strict order relation.
- A relative semi-ordinal is a semi-ordinal equipped with a wide sub-semi-category.
- Kock defined $\underline{\Delta}$ as the category of finite non-empty relative semi-ordinal and marking-preserving functors.
- Full and faithful functor $\mathrm{rso} : \underline{\Delta} \hookrightarrow \mathrm{RelSemiCat}$ where RelSemiCat denotes relative semi-categories and marking-preserving functors.

Example

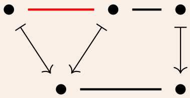
The relative semi-ordinals

$$\bullet \qquad \bullet \longrightarrow \bullet \qquad\qquad \bullet \longrightarrow \bullet \qquad \bullet$$

correspond to the epimorphisms



and



Horizontal and vertical inclusions

- The horizontal inclusion

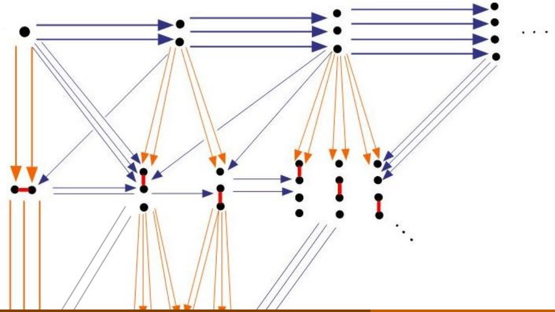
$$(\_)^\flat : \Delta_+ \hookrightarrow \underline{\Delta},$$

  maps $[n]$ to $\mathrm{id}_{[n]}$. The image of the horizontal inclusion corresponds to relative semi-ordinals with nothing marked.

- The vertical inclusion

$$(\_)^\sharp : \Delta_+ \hookrightarrow \underline{\Delta}$$

  maps a semi-ordinal $[n]$ to $[n] \twoheadrightarrow [0]$. The objects in the image of $(\_)^\sharp$ correspond to the relative semi-ordinals with everything marked.



Over to Simona.

Fat delta via monads with arities: overview

- The theory of monads with arities, originated in the work of Berger, Melliès, and Weber, was applied to a number of categories that proved fundamental in higher category theory.
- They include Δ, Joyal's disk category Θ_n, the dendroidal category etc.
- Applying the theory of monad with arities to fat Delta, we will

  - prove a nerve theorem for relative semicategories;
  - show that fat Delta is a hypermoment category.

Local right adjoints

- A functor $R : \mathcal{E} \to \mathcal{F}$ is said to be a *local right adjoint* if the induced functor $R_X : \mathcal{E}/X \to \mathcal{F}/RX$ has a left adjoint L_X, for each object X.

- A morphism $g : A \nrightarrow RX$ is said to be R-generic if, given α, β and γ making the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "g"', "|" marking] & RX' \arrow[d, "R\gamma"] \\ RX \arrow[r, "R\beta"'] \arrow[ur, dotted, "R\delta" description] & RY \end{tikzcd}$$

there is a unique $\delta : X \to X'$ such that $R\delta g = \alpha$ and $\beta = \gamma\delta$.

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\end{tikzcd}$$

  there is a unique $\delta : X \to X'$ such that $R\delta g = \alpha$ and $\beta = \gamma\delta$.

Proposition

For a functor $R : \mathcal{E} \to \mathcal{F}$ where $\mathcal{E}$ has a terminal object, the following are equivalent:

- *R is a local right adjoint.*
- *the functor $R_1 : \mathcal{E} \to \mathcal{F}/R1$ has a left adjoint;*
- *each $f : A \to R1$ factors as $A \overset{g}{\nvrightarrow} RX \xrightarrow{R!_X} R1$, where g is R-generic.*

Dense generators

- Let $\mathcal{E}$ be a category. A full subcategory $\mathcal{A}$ and the inclusion functor $i_{\mathcal{A}} : \mathcal{A} \to \mathcal{E}$ are called *dense* if $\mathcal{A}$ is small and the associated *nerve functor*

$$\mathcal{N}_{\mathcal{A}} : \mathcal{E} \to \widehat{\mathcal{A}}, \quad x \mapsto \mathcal{E}(i_{\mathcal{A}}, x)$$

  is fully faithful. The dense subcategory $\mathcal{A}$ is also called a *dense generator*.

- Density is equivalent to requiring that any object X in $\mathcal{E}$ is the colimit over the comma category $\mathcal{A}/X$ given by the composition $i_{\mathcal{A}}\pi_X$, where $\pi_X : \mathcal{A}/X \to \mathcal{A}$ is the domain projection.

- Let us call such colimits *canonical $\mathcal{A}$-colimits*.

Monad with arities

- A monad (T, μ, ν) on a category $\mathcal{E}$ with a dense generator $i_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{E}$ is called a *monad with arities* if the composition $\mathcal{N}_{\mathcal{A}} T$ preserves canonical $\mathcal{A}$-colimits.
- Denote by $EM(T)$ the Eilenberg-Moore category consisting of T-algebras (X, α), $\alpha : TX \to X$ with $\alpha T(\alpha) = \alpha\mu_X$, $\alpha\nu_X = \mathrm{Id}_X$ and morphisms commuting with the T-algebra structure.
- Let Θ_T be obtained by factoring
  $$F^T i_{\mathcal{A}} : \mathcal{A} \to EM(T)$$
  into bijective on objects followed by fully faithful $\mathcal{A} \xrightarrow{j_T} \Theta_T \xrightarrow{i_T} EM(T)$, so Θ_T is the full subcategory spanned by the free T-algebras over $\mathcal{A}$

Nerve Theorem

Theorem (Berger, Melliès, Weber 2012)

Let T be a monad with arities $\mathcal{A}$. The nerve functor $\mathcal{N}_T : EM(T) \to \hat{\Theta}_T$ associated to $i_T : \Theta_T \to EM(T)$ is fully faithful and its essential image is spanned by the presheaves whose restriction along $j_T : \mathcal{A} \to \Theta_T$ belongs to the essential image of $\mathcal{N}_{\mathcal{A}} : \mathcal{E} \to \widehat{\mathcal{A}}$.

Canonical arities

- A monad is *strongly cartesian* if it is cartesian and a local right adjoint.
- Let T be a strongly cartesian monad on a finitely complete category $\mathcal{E}$ with a dense generator $i_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{E}$. The category of canonical arities $Ar_{\mathcal{A}}(T)$ is the essential image of the composition

$$\mathcal{A} / T1 \xrightarrow{i_A / T1} \mathcal{E} / T1 \xrightarrow{L_1} \mathcal{E}$$

where $L_1 : \mathcal{E} / T1 \to \mathcal{E}$ is left adjoint to $T_1 : \mathcal{E} \to \mathcal{E} / T1$.

- A dense generator $\mathcal{A}$ of $\mathcal{E}$ is T-generically closed if for any T-generic morphism $A \to TB$ with A in $\mathcal{A}$, there is an object isomorphic to B which belongs to $\mathcal{A}$.

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$$\mathcal{A}\,/\,T1 \xrightarrow{i_A/T1} \mathcal{E}\,/\,T1 \xrightarrow{L_1} \mathcal{E}$$

where $L_1 : \mathcal{E}\,/\,T1 \to \mathcal{E}$ is left adjoint to $T_1 : \mathcal{E} \to \mathcal{E}\,/\,T1$.

- A dense generator $\mathcal{A}$ of $\mathcal{E}$ is T-generically closed if for any T-generic morphism $A \to TB$ with A in $\mathcal{A}$, there is an object isomorphic to B which belongs to $\mathcal{A}$.

Theorem (Berger, Melliès, Weber 2012)

Suppose T is a strongly cartesian monad on $\mathcal{E}$ with a dense generator $\mathcal{A}$. Then the category of canonical arities $Ar_{\mathcal{A}}(T)$ is T-generically closed and provides T with arities $Ar_{\mathcal{A}}(T)$.

Example

- $T : \mathsf{Gph} \to \mathsf{Gph}$ free category monad on graphs.
- Arities: Δ_0 (subcategory of distance-preserving maps)
- $\Theta_T : \Delta$.
- Nerve functor $\mathcal{N} : \mathsf{Cat} \to \hat{\Delta}$ is fully faithful.
- Essential image: simplicial sets satisfying Segal condition.

The free semi-category monad

- There is a free-forgetful adjunction

$$F : \mathsf{Gph} \rightleftarrows \mathsf{seCat} : U$$

  between graphs and semi-categories. FX is the path semi-category of X.

- The corresponding monad $\mathtt{f} : \mathsf{Gph} \to \mathsf{Gph}$ is the free relative semi-category monad.

Proposition

The monad $\mathtt{f} : \mathsf{Gph} \to \mathsf{Gph}$ *is strongly cartesian.*

The free relative semi-category monad

- There is a free-forgetful adjunction

$$F^r : \mathsf{Gph}^{\mathsf{r}} \rightleftarrows \mathsf{seCat}^{\mathsf{r}} : U^r$$

  where $\mathsf{Gph}^{\mathsf{r}}$ is the category of relative graphs.

- The corresponding monad $\mathtt{f}^r : \mathsf{Gph}^{\mathsf{r}} \to \mathsf{Gph}^{\mathsf{r}}$ is the free relative semi-category monad.

Proposition

The monad $\mathtt{f}^r : \mathsf{Gph}^{\mathsf{r}} \to \mathsf{Gph}^{\mathsf{r}}$ *is strongly cartesian.*

Arities of the free relative semi-category monad

- We can provide arities to the free relative semi-category monad $\mathtt{f}^r$ by fixing a dense generator of Gph^r.

- If we take the smallest dense generator, generated by $[0], [1]^b, [1]^\sharp$, the canonical arities correspond to the full subcategory of linearly ordered graphs that are fully marked or unmarked.

- To capture the whole of $\underline{\Delta}$ we refine the dense generator to include the partially marked objects.

Arities of the free relative semi-category monad, cont.

- Let $\mathcal{A}$ be the full subcategory of linearly ordered graphs with no two adjacent edges of the same markings; for instance:

$$\langle \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \rangle,$$
$$\langle \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \rangle$$

- The category of canonical arities $Ar_{\mathcal{A}}(\mathtt{f}^r)$ is the full subcategory of partially marked linear graphs.

Theorem (De Jong, K., P., Pradal)

The nerve functor $\mathcal{N} : \mathsf{seCat}^\mathsf{r} \to \hat{\underline{\Delta}}$ *is fully faithful. Its essential image is spanned by the presheaves whose restriction along* $j : \underline{\Delta}_0 \to \underline{\Delta}$ *belong to the essential image of* $\mathcal{N}_0 : \mathsf{Gph}^\mathsf{r} \to \hat{\underline{\Delta}}_0$.

Hypermoment categories

- Hypermoment categories are a general setting to study operadic structures.
- Given a hypermoment category $\mathbb{C}$, there are notions of $\mathbb{C}$-operads and $\mathbb{C}_\infty$-monoids,
- Examples
    - $\mathbb{C} = \Gamma$, symmetric operads, E_∞-spaces.
    - $\mathbb{C} = \Delta$, non-symmetric operads, A_∞-spaces.
    - $\mathbb{C} = \Theta_n$, n-operads, E_n-spaces.

Hypermoment categories: definition

Definition (Berger)

A hypermoment category $\mathbb{C}$ is a category equipped with an orthogonal factorization system (called active-inert) and a map $\gamma_{\mathbb{C}} : \mathbb{C} \to \Gamma$ such that

i) γ preserves active (resp. inert) morphisms.

ii) For each object A of $\mathbb{C}$ and each morphism $\{0\} \to \gamma_{\mathbb{C}}(A)$ there is an essential unique inert lift $U \to A$ in $\mathbb{C}$ where U is such each active morphism with target U has one and only one inert section.

Fat delta as an hypermoment category

Theorem (De Jong, K., P., Pradal)

The category $\underline{\Delta}$ is a hypermoment category which is strongly unital and extensional.

- Here, strong unitality allows to have a notion of $\underline{\Delta}$-operads.
- Extensionality is a property that could led to a notion of $\underline{\Delta}$-decomposition space.

Active-inert factorization system in $\underline{\Delta}$

The classes $\underline{\Delta}_a$ and $\underline{\Delta}_0$ of active and inert maps in $\underline{\Delta}$ are given by

$$\begin{tikzcd}
{[m]} \arrow[r, hook, "active"'] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner", very near end] & {[m']} \arrow[d, two heads] & {[m]} \arrow[r, hook, "inert"] \arrow[d, two heads] & {[m']} \arrow[d, two heads] \\
{[n]} \arrow[r] & {[n']} & {[n]} \arrow[r] & {[n']}
\end{tikzcd}$$

where an active map m in Δ is end-point preserving and inert map m in Δ is distance preserving.

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