Generalized Decidability via Brouwer Trees

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Summary Decidability and semidecidability are at the heart of computer science. If we think of "decidable" as "the answer is discoverable in finitely many (i.e. less than ω, the smallest infinite ordinal) steps," then "semidecidable" means that the answer is discoverable in less than $\omega + 1$ steps. This formulation allows for a finer hierarchy than just "decidable, semidecidable, undecidable." For example, if every P_i is semidecidable, then $\forall i \in \mathbb{N}.P_i$ may not be semidecidable, but intuitively, its answer is discoverable in ω^2 steps: each $P(i)$ requires at most ω steps, and there are ω instances. An example is the twin prime conjecture: For every n, it is semidecidable whether there is a pair $(p, p + 2)$ of primes larger than n, but asking whether this is true for all n would, at least on an intuitive level, require ω^2 steps. We suggest and study a framework in which this and related statements can be formulated and proved. This abstract is based on our paper [5] which has been accepted at *LICS 2026* and is available on arXiv. All our results are formalized in cubical Agda [10] and the formalization can be found in our Git repository [6].

A preliminary variant of Definition 1 below, and its relation to semidecidability, were presented in a talk at TYPES'22 [8]. Here, we develop and refine the theory suggested in that talk. For transparency, we also declare that the current talk proposal has been accepted for presentation at TYPES'26; in comparison, we hope that we will be able to spend more time on the construction of the characteristic function and countable choice (see below) at HoTT/UF'26.

Setting and Background We work in homotopy type theory, following notions of decidability in constructive mathematics. A proposition P is decidable if $P \vee \neg P$ holds, and semidecidable if there exists a binary sequence that is 1 somewhere if and only if P holds — see the work on synthetic computability theory by Bauer [1] and Forster, Kirst, and Smolka [3]. The type Brw of *Brouwer tree ordinals* (in the following just *ordinals*) is given by a quotient inductive-inductive construction together with an order relation $\leq\ : \mathsf{Brw} \to \mathsf{Brw} \to \mathsf{Prop}$ [9]. The constructors for Brw are zero, succ, and limit, constructing zero, successors, and limits of sequences (cf. Kleene's $\mathcal{O}$ [7, 2]), together with a path constructor to ensure that bisimilar sequences give equal limits, and constructors for the relation $\leq$ to ensure that the expected equations holds. An important design choice for the current work is that the constructor limit only accepts *strictly* increasing sequences as its argument. As a consequence, it is decidable whether a given ordinal is finite or infinite: zero is finite, $\mathsf{limit}\, f$ is always infinite, and $\mathsf{succ}\, \beta$ is finite if and only if β is; moreover, each ordinal α can be written as $\lambda + n$, where λ is a limit (or zero) and n is a natural number.

Generalized Decidability

Definition 1. Let $\alpha : \mathsf{Brw}$ be an ordinal and P a proposition. We say that P is α-decidable if there exists $x : \mathsf{Brw}$ such that $P \leftrightarrow x \geq \alpha$.

For sufficiently low ordinals α, the statement "P is α-decidable" can be understood intuitively as "if evidence for P exists, then it can be discovered in less than α steps," as discussed in the first paragraph. One can show that P is decidable if and only if it is ω-decidable, i.e. evidence can be discovered in finitely many steps, and semidecidable if and only if its evidence can be discovered in less than $(\omega + 1)$ steps.

Remark 2. For the technical development, it is useful to observe that considering limit ordinals is sufficient; for example, the notions of $(\omega + 1)$-decidability and $(\omega \cdot 2)$-decidability coincide, and the first steps of the hierarchy can be presented as follows:

$$\begin{aligned} P \text{ holds} &\leftrightarrow P \text{ is } (\omega \cdot 0)\text{-decidable} \\ P \text{ is decidable} &\leftrightarrow P \text{ is } (\omega \cdot 1)\text{-decidable} \\ P \text{ is semidecidable} &\leftrightarrow P \text{ is } (\omega \cdot 2)\text{-decidable} \end{aligned}$$

A proof of this is given in [5, Proposition 3.2].

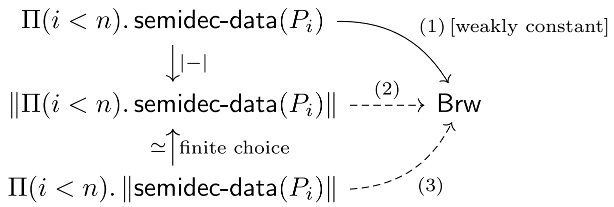
Closure under Logical Connectives The decidable propositions and the semidecidable propositions are closed under binary conjunctions as well as binary disjunctions. The more general situation is subtle.

Let P and Q be α-decidable, evidenced by x and y (i.e. $P \leftrightarrow x \geq \alpha$ and $Q \leftrightarrow y \geq \alpha$). If m is the minimum of the two numbers (i.e., $\forall \gamma.\, m \geq \gamma \leftrightarrow (x \geq \gamma \times y \geq \gamma)$) then m is evidence that $P \times Q$ is α-decidable. In general, we cannot expect to be able to calculate such a minimum, as this would imply the Limited Principle of Omniscience (LPO), a constructive taboo. Fortunately, we are able to define a function $\mathsf{limMin} : \mathsf{Brw} \to \mathsf{Brw} \to \mathsf{Brw}$ that is correct as long as the two arguments are limits, which suffices in line with Remark 2. On the formalization side, it turned out that a convenient technique is a relational approach to the limMin function.

Binary disjunctions would follow similarly if we could calculate binary maxima of limits. However, and possibly surprisingly, the existence of binary maxima would imply LPO even if restricted to limits. Therefore, we only get a restricted form of closure under binary disjunctions:

Theorem 3. The α-decidable propositions are closed under binary conjunction: if P and Q are α-decidable, then so is $P \times Q$. For natural numbers n and k, the $(\omega \cdot n + k)$-decidable propositions are closed under binary disjunctions: if P and Q are $(\omega \cdot n + k)$-decidable, then so is $\|P + Q\|$. □

Quantifiers Assume that $P : \mathbb{N} \to \mathsf{Prop}$ is a family of α-decidable propositions; our next goal is to characterize the decidability of $\forall i.\, P_i$ and $\exists i.\, P_i$. For the time being, we only have results for families of semidecidable propositions ($\alpha = \omega \cdot 2$), which covers the above-mentioned twin prime conjecture. To solve this case, we calculate the *characteristic ordinal* of P, with the following strategy. In the first step (1), we define a function $\mathbb{N} \to \mathsf{Brw}$ which, at position n, assumes that the binary sequence evidencing semidecidability is given as data (Σ instead of $\exists$) for all P_i where $i < n$. The second step (2) is to show that this function is (weakly) constant in the semidecidability data, and thus is well-defined even if the evidence for semidecidability merely exists. The third and final step (3) uses finite choice (i.e. the repeated application of the equivalence $\|A \times B\| \simeq \|A\| \times \|B\|$). These steps are illustrated and summarized in the diagram below, where we have fixed $n : \mathbb{N}$.



The map (3) in the above construction yields a function $\Psi : \mathbb{N} \to \mathsf{Brw}$. The characteristic ordinal of P is then defined as $\mathsf{limit}(n \mapsto \Psi_n + n)$, where the part $+n$ ensures that the sequence is strictly increasing.

The characteristic ordinal is used to prove the following:

Theorem 4. If $P : \mathbb{N} \to \mathsf{Prop}$ is a family of semidecidable propositions, then $\forall i.\, P_i$ is ω^2-decidable, and $\exists i.\, P_i$ is $(\omega \cdot 3)$-decidable. □

The characteristic ordinal also allows us to show that, if $Q : \mathbb{N} \times \mathbb{N} \to \mathsf{Prop}$ is a family of semidecidables such that $Q(n, m)$ implies $Q(n, m+1)$, then $\exists m.\, \forall n.\, Q(n, m)$ is $(\omega^2 + \omega)$-decidable. An instance is the search for a counter-example for the twin prime conjecture.

Generalized Decidability and Countable Choice As we consider sequences indexed by $\mathbb{N}$, it is unsurprising that the assumption of countable choice makes a difference. It is well-known that, under countable choice, semidecidable propositions are closed under countable joins, in contrast to our Theorem 4. We generalize this fact:

Theorem 5. Assuming countable choice, every $(\omega \cdot k)$-decidable proposition is semidecidable. □

However, countable choice does not collapse the whole hierarchy, as ω^2-decidable propositions are distinguishable from $(\omega \cdot k)$-decidable ones. There are models where both countable choice and Markov's principle hold, but LPO does not [4], which means that the following result shows that countable choice is not sufficient to prove that ω^2-decidable propositions are semidecidable.

Theorem 6. If every ω^2-decidable proposition is semidecidable, then MP implies LPO. □

Consequences for ω^2-decidable proposition include the following:

Theorem 7. Assuming countable choice, any ω^2-decidable proposition is the countable meet of semidecidables. Further, ω^2-decidable propositions are closed under countable meets. □

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