Ordinal Exponentiation in Homotopy Type Theory

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Logic in Computer Science

24 June 2025

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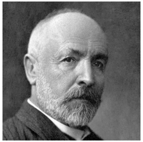
What are ordinal numbers?

- They generalize $\mathbb{N}$: $0,\ 1,\ 2, \dots,\ \omega,\ \omega{+}1,\ \omega{+}2, \dots,\ \omega^{\omega}, \dots$
- Key property: Every decreasing sequence terminates.
- Applications: prove termination of processes, show consistency of logical theories, justify induction and recursion principles, ...



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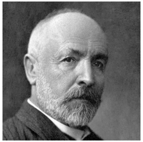

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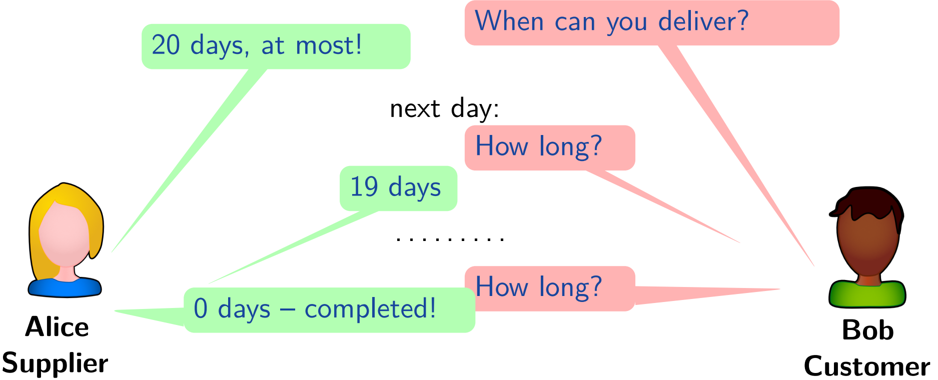

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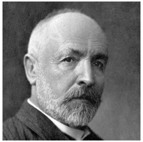

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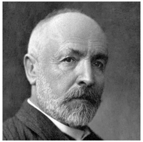

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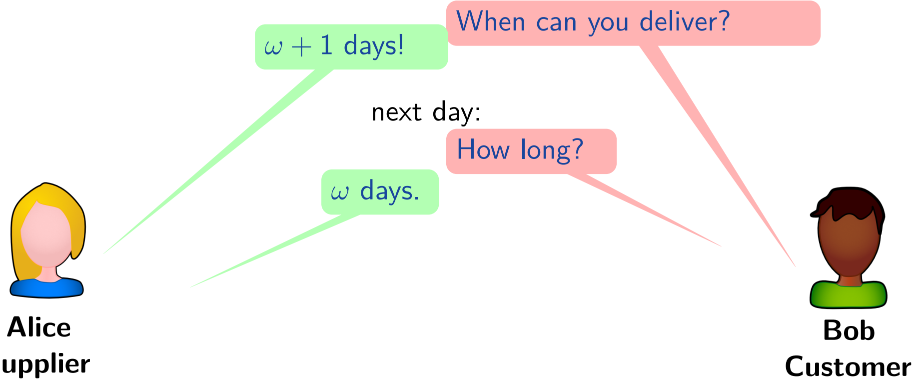

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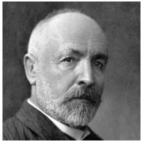

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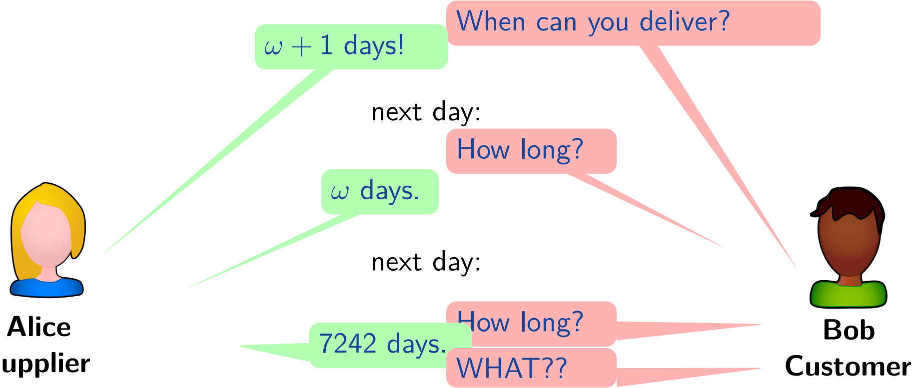
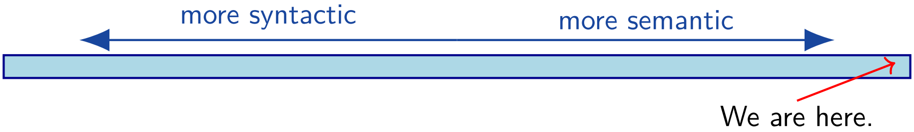


Our goal: constructive ordinal theory

- **We want a constructive version of the theory of ordinals.**
- Classically, ordinals have many different representations;
  but constructively, the notions split apart:



- For us, an ordinal is a set α with a relation $<$ which is transitive, extensional, wellfounded.
  This notion was previously suggested by Grayson (1978, 1982) and was studied in the HoTT book (2013) as well as by Escardó (2020s).

We want arithmetic

- Classically, arithmetic operations are usually defined by **inspecting whether the exponent is zero, a successor, or a limit ordinal**.
- For example, addition:

$$\begin{aligned} \alpha + 0 &:= \alpha \\ \alpha + (\beta + 1) &:= (\alpha + \beta) + 1 \\ \alpha + \lambda &:= \sup_{\beta<\lambda}(\alpha + \beta) \qquad\qquad \text{(if } \lambda \text{ is a limit)} \end{aligned}$$

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- Such a case distinction is not possible when working constructively, so this is not a definition — but it can still serve as a specification.
- There is an easy and well-known constructive solution: $\alpha + \beta \ := \ \alpha \uplus \beta$ (the disjoint union), with everything from α being below everything from β.

The problem with exponentiation

- Classically:

$$\alpha^0 = 1 \qquad\qquad 0^\beta = 0 \qquad\qquad (\text{if } \beta \neq 0)$$

$$\alpha^{\beta+1} = \alpha^\beta \times \alpha \qquad\qquad \alpha^\lambda = \sup_{\beta<\lambda} \alpha^\beta \quad (\text{if } \lambda \text{ is a limit, } \alpha \neq 0)$$

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\end{aligned}
$$

- No easy constructive solution; in fact:
  Thm. (Imprecise) There is a well behaved exponentiation function on *all* ordinals if and only if excluded middle holds.
- Exponentiation is unintuitive, e.g.

$$2^\omega \; = \; 2^{\sup(0,1,2,\ldots)} \; = \; \sup(2^0, 2^1, 2^2, \ldots) \; = \; \omega$$

- In particular, exponentiation is *not* given by function spaces.

Our work – the highlights

- Working constructively in homotopy type theory, we construct two well behaved ordinal exponentiation functions $\alpha^{(-)}$ with a minor condition on the base α:

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    It is well defined whenever $\alpha \geq \mathbf{1}$, i.e. whenever α has a least element.
  - The second is more concrete, **based on decreasing lists**, and a constructive version of a construction by Sierpiński based on functions with finite support.

    It is well defined whenever α has a trichotomous least element: the least element $a_\perp \in \alpha$ is further required to satisfy: $\forall (x \in \alpha).\ (a_\perp < x) \vee (a_\perp = x)$.

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- We show that our two constructions agree (whenever the base ordinal has a trichotomous least element).

Commercial break

- A link to our paper is in the programme.

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***Abstract*—We present two seemingly different definitions of constructive ordinal exponentiation, where an ordinal is taken to be a transitive, extensional, and wellfounded order on a set. The first definition is abstract, uses suprema of ordinals, and is solely motivated by the expected equations. The second is more concrete, based on decreasing lists, and can be seen as a constructive version of a classical construction by Sierpiński based on functions with finite support. We show that our two approaches are equivalent (whenever it makes sense to ask the question), and use this equivalence to prove algebraic laws and decidability properties of the exponential. Our work takes place in the framework of homotopy type theory, and all results are formalized in the proof assistant Agda.**

I. Introduction

In classical mathematics and set theory, ordinals have rich and interesting structure. How much of this structure can be developed in a constructive setting, such as homotopy type theory? This is not merely a question of mathematical curiosity, as classical ordinals have powerful applications as tools for establishing consistency of logical theories [1], proving termination of processes [2], and justifying induction and recursion [3], [4], which would all be valuable to have available in constructive mathematics and proof assistants based on constructive type theory. There are many constructive approaches to ordinals, such as ordinal notation systems [5], Brouwer trees [6], or wellfounded trees with finite or countable branchings [7], [8], to name a few. In this paper, we follow the Homotopy Type Theory Book [9] and consider ordinals as order types of well ordered sets, i.e., an ordinal is a type equipped with an order relation having certain properties.

Ordinals have an arithmetic theory that generalizes the one of the natural numbers. Classically, arithmetic operations are defined by case distinction and transfinite recursion. For addition and multiplication, we have:

$$\begin{aligned} \alpha + 0 &= \alpha \\ \alpha + (\beta + 1) &= (\alpha + \beta) + 1 \\ \alpha + \lambda &= \sup_{\beta<\lambda} (\alpha + \beta) && \text{(if } \lambda \text{ is a limit)} \\ \alpha \times 0 &= 0 \\ \alpha \times (\beta + 1) &= (\alpha \times \beta) + \alpha \\ \alpha \times \lambda &= \sup_{\beta<\lambda} (\alpha \times \beta) && \text{(if } \lambda \text{ is a limit)} \end{aligned}$$

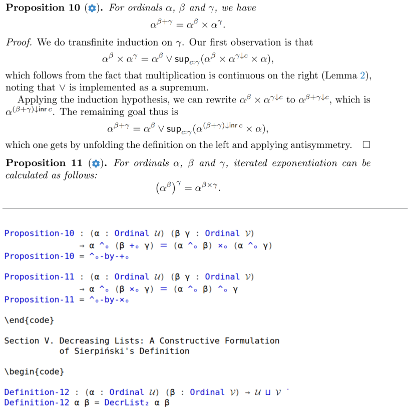
Similarly, exponentiation is classically defined by:

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta \times \alpha \\ \alpha^\lambda &= \sup_{\beta<\lambda} \alpha^\beta && \text{(if } \lambda \text{ is a limit, } \alpha \neq 0) \\ 0^\beta &= 0 && \text{(if } \beta \neq 0) \end{aligned} \tag{†}$$

We can recognize the equations from arithmetic on the natural numbers in the cases of 0 and successors $\beta+1$, extended by a case for limit ordinals, which ensures that each operation is continuous in the second argument. However, while the situation might seem familiar, it is in fact constructively very different: for general well orders, such a case distinction on whether an ordinal is zero, a successor, or a limit, is available if and only if the law of excluded middle holds. Constructively, the above equations are thus not a valid definition of the arithmetic operations on ordinals. However, they are still a reasonable *specification* of how the operations should behave.

For addition and multiplication, there are well known alternative constructions based on a clear visual intuition, using disjoint unions and Cartesian products of order types, respectively (cf. the related work paragraph below and especially Escardó's work [10]). These are well behaved constructively — for example, one can prove that they satisfy the specification on the left. On the other hand, subtraction (and, similarly, division) is inherently non-constructive for the approach of ordinals as well ordered sets [10, Ordinals.AdditionProperties]. For exponentiation, most classical textbooks on ordinals [11], [12], [13], [14], [15], [16] simply employ the non-constructive definition by cases. Notable exceptions are the classical monograph by Sierpiński [17] and the constructive work by Grayson [18], [19], which we will return to shortly. In fact, going back all the way to Cantor's original writings on ordinals [20], addition and multiplication are defined "explicitly" in §14, whereas exponentiation (with base $\alpha > 1$) is defined by case distinction and transfinite recursion in §18.

Perhaps one reason why constructive ordinal exponentiation has so far been underdeveloped is that the operation is somewhat non-intuitive, even in a classical setting. For example, for the first infinite ordinal $\omega = \sup_{n:\mathbb{N}} n$, continuity in the exponent (cf. the specification (†)) implies that $2^\omega = \omega$ — an equation very different from what one might expect from, for example, cardinal exponentiation. This in turn rules out an understanding of ordinal exponentiation in terms of category



Proposition 10 (⚙). *For ordinals α, β and γ, we have*

$$\alpha^{\beta+\gamma} = \alpha^\beta \times \alpha^\gamma.$$

Proof. We do transfinite induction on γ. Our first observation is that

$$\alpha^\beta \times \alpha^\gamma = \alpha^\beta \vee \sup_{c:\gamma}(\alpha^\beta \times \alpha^{\gamma\downarrow c} \times \alpha),$$

which follows from the fact that multiplication is continuous on the right (Lemma 2), noting that $\vee$ is implemented as a supremum.

Applying the induction hypothesis, we can rewrite $\alpha^\beta \times \alpha^{\gamma\downarrow c}$ to $\alpha^{\beta+\gamma\downarrow c}$, which is $\alpha^{(\beta+\gamma)\downarrow \mathsf{inr}\, c}$. The remaining goal thus is

$$\alpha^{\beta+\gamma} = \alpha^\beta \vee \sup_{c:\gamma}(\alpha^{(\beta+\gamma)\downarrow \mathsf{inr}\, c} \times \alpha),$$

which one gets by unfolding the definition on the left and applying antisymmetry. □

Proposition 11 (⚙). *For ordinals α, β and γ, iterated exponentiation can be calculated as follows:*

$$(\alpha^\beta)^\gamma = \alpha^{\beta\times\gamma}.$$

```
Proposition-10 : (α : Ordinal 𝓤) (β γ : Ordinal 𝓥)
               → α ^ₒ (β +ₒ γ) ＝ (α ^ₒ β) ×ₒ (α ^ₒ γ)
Proposition-10 = ^ₒ-by-+ₒ

Proposition-11 : (α : Ordinal 𝓤) (β γ : Ordinal 𝓥)
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Proposition-11 = ^ₒ-by-×ₒ

\end{code}

Section V. Decreasing Lists: A Constructive Formulation
           of Sierpiński's Definition

\begin{code}

Definition-12 : (α : Ordinal 𝓤) (β : Ordinal 𝓥) → 𝓤 ⊔ 𝓥 ̇
Definition-12 α β = DecrList₂ α β
```

- All of its results are formalized in the proof assistant **Agda**, based on Martín Escardó's library **TypeTopology**. Clicking a ⚙ next to a definition, lemma, theorem, etc. in the paper takes you to its formalization.

First solution: abstract exponentiation

- Our no-go theorem says that we cannot define α^β for arbitrary α, β. Let's assume $\alpha \geq 1$. We can shorten the specification to:

$$\alpha^{\beta+\mathbf{1}} = \alpha^\beta \times \alpha \qquad\qquad \alpha^{\sup_{i \in I} F_i} = \mathbf{1} \vee \sup_{i \in I}(\alpha^{F_i})$$

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- If β is an ordinal and $b_0 \in \beta$, then the initial segment $\beta{\downarrow}b_0$ is an ordinal:

$$\beta{\downarrow}b_0 \;:=\; \{b \in \beta \mid b < b_0\}$$

- <u>Lemma</u>. For every ordinal β we have $\beta = \sup_{b_0\in\beta}(\beta{\downarrow}b_0 + \mathbf{1})$.

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$$\beta\downarrow b_0 := \{b \in \beta \mid b < b_0\}$$

- Lemma. For every ordinal β we have $\beta = \sup_{b_0\in\beta}(\beta\downarrow b_0 + \mathbf{1})$.
- **Idea**: *If* we had α^β, then

$$\alpha^\beta = \alpha^{\sup_{b:\beta}(\beta\downarrow b+\mathbf{1})} = \mathbf{1} \vee \sup_{b:\beta} \alpha^{\beta\downarrow b+\mathbf{1}} = \mathbf{1} \vee \sup_{b:\beta}\left(\alpha^{\beta\downarrow b} \times \alpha\right).$$

$\Rightarrow$ We take this as a definition (called abstract exponentiation).

Second solution: concrete exponentiation

- Sierpiński (1958):

$$\alpha^{\beta} := \{f : \alpha \to \beta \mid f \text{ has finite support (i.e. is almost everywhere 0)}\}$$

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- Constructively well behaved version: represent a function with finite support as a **list of (output, input) pairs**. The least element $a_{\perp} \in \alpha$ should not be an output, so we consider

$$\alpha_{>\perp} := \{a \in \alpha \mid a > a_{\perp}\}$$

  and define concrete exponentiation $\exp(\alpha, \beta)$ as the set of lists on $\alpha_{>\perp} \times \beta$ that are decreasing in the β-component:

$$\exp(\alpha, \beta) := \{[(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)] \mid a_i \in \alpha_{>\perp}, b_i \in \beta, b_1 > b_2 > \ldots > b_k\}$$

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- We require a **trichotomous least element** $a_\perp$, i.e. $a_\perp$ satisfies $\forall x \in \alpha.\ (a_\perp < x) \vee (a_\perp = x)$, to ensure that $\alpha_{>\perp}$ is an ordinal.

The whole is greater than the sum of its parts

- Thm. Concrete and abstract exponentiation coincide (as long as $a_\perp$ is trichotomous).
  $\Rightarrow$ We get the best of both worlds.
- For example, for the abstract exponentiation, it is easy to show:

$$\alpha^{\beta+\gamma} = \alpha^{\beta} \times \alpha^{\gamma} \quad \text{and} \quad \alpha^{\beta\times\gamma} = \left(\alpha^{\beta}\right)^{\gamma}.$$

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Thanks for your attention!