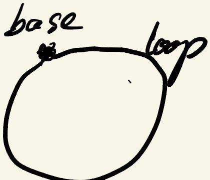
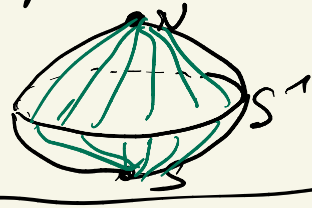
Calculating $base = base$ in S^1

data S^1 where
  $base : S^1$
  $loop : base = base$



topology:
fund. group of circle $\simeq \mathbb{Z}$

data S^2 where
  $N : S^2$
  $S : S^2$
  $p : S^1 \to N = S$



Goal: Show $(base = base) \simeq \mathbb{Z}$ in HoTT

Define $\mathbb{Z}$ as $\underbrace{Nat}_{0,1,2,3\ldots} \uplus \underbrace{Nat}_{-1,-2,\ldots}$

$in1 : Nat \to \mathbb{Z}$
$in2 : Nat \to \mathbb{Z}$

$in1\ n$ stands for "$+n$"
$in2\ n$ stands for "$-(n+1)$"

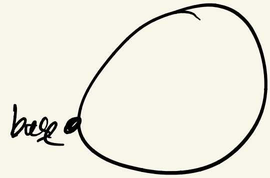
$wind : \mathbb{Z} \to base = base$
(aka $loop^-$)

$$wind\ n :\equiv \underbrace{loop \overset{trans}{\cdot} loop \cdot \ldots \cdot loop}_{n \text{ times}}$$

$wind\ (in1\ 0) :\equiv refl$
$wind\ (in1\ Sk) :\equiv wind\ (in1\ k) \cdot loop$
$wind\ (in2\ 0) :\equiv loop^{-1}$ [aka sym loop]
$wind\ (in2\ Sk) :\equiv wind\ (in2\ k) \cdot loop^{-1}$

$\text{unwind} : \text{base} = \text{base} \to \mathbb{Z}$

$\text{unwind}\ \ p \ :\equiv \mathbb{Z}$



define $\text{Code} : S^1 \to \text{Type}$

$\text{Code}(\text{base}) :\equiv \mathbb{Z}$

or: $\text{Code}(\text{loop})$

$\text{ap}_{\text{Code}}(\text{loop}) :\equiv \text{ua}(\text{suc}, e)$

$\boxed{\text{ap also called cong}}$

$(\text{suc}, e) : \mathbb{Z} \simeq \mathbb{Z}$ $\qquad \text{suc} : \mathbb{Z} \to \mathbb{Z}$

$\text{ua}(\text{suc}, e) : \mathbb{Z} = \mathbb{Z}$ $\qquad e : \text{isEqv}(\text{suc})$

$\text{ap}_{\text{Code}}(\text{loop}) : \mathbb{Z} = \mathbb{Z}$

recall: $\text{elim}_{S^1} : (A : \text{Type}) \to (a_0 : A) \to (a_0 = a_0)$
$\to S^1 \to A$

$\text{code} :\equiv \text{elim}_{S^1}\ \text{Type}\ \mathbb{Z}\ (\text{ua}(\text{suc}, e))$

$\text{ap}_{\text{Code}}^{\{\text{base}\}\{\text{base}\}} : (\text{base} = \text{base}) \to \mathbb{Z} = \mathbb{Z}$

$\text{id2eqv} : A = B \to A \simeq B$

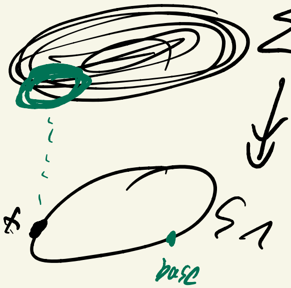
$\text{proj} : A \simeq B \to (A \to B)$

$\text{unwind} : \text{base} = \text{base} \to \mathbb{Z}$

$\text{unwind}\ \ p :\equiv \text{proj}(\text{id2eqv}(\text{ap}_{\text{Code}}(p)))(0)$ in

$\mathrm{Code} : S^1 \to \mathrm{Type}$

$\Sigma(x:S^1).\,\mathrm{Code}(x)$



Goal: wind & unwind are inverse

easy case: $(n:\mathbb{Z}) \to \mathrm{unwind}\,(\mathrm{wind}\, n) = n$

$$\mathrm{unwind}\,(\mathrm{wind}\,(\mathrm{int}\ 0))$$
$$= \mathrm{unwind}(\ \mathrm{refl}\ )$$
$$= \mathrm{int}\ 0$$

$$\mathrm{unwind}\,(\mathrm{wind}\,(\mathrm{int}\ Sk))$$
$$= \mathrm{unwind}\,(\mathrm{wind}\,(\mathrm{int}\ k) \cdot \mathrm{loop})$$

[lemma: $\mathrm{ap}_f(p \cdot q) = \mathrm{ap}_f\, p \cdot \mathrm{ap}_f\, q$]

$$= \mathrm{proj}(\mathrm{id2equ}(\mathrm{ap}_{\mathrm{Code}}(\mathrm{int}\ k) \cdot \mathrm{ap}_{\mathrm{Code}}(\mathrm{loop})))(0)$$
$$= (\mathrm{calc.}) = \mathrm{int}\,(Sk)$$

difficult direction:

$$(p : base = base) \to wind\ (unwind\ p) = p$$

generalise:

$$\text{unwind-gen} : (x : S^1) \to (base = x) \to Code\ x$$
$$\text{wind-gen} : (x : S^1) \to Code\ x \to base = x$$

then it becomes easy:

$$(x : S^1) \to (p : base = x) \to \text{wind-gen}\ (\text{unwind-gen}\ p) = p$$

$$\begin{aligned} & \text{wind-gen}\ (\text{unwind-gen}\ refl) \\ &= \text{wind-gen}\ (unwind\ refl) \\ &= \text{wind-gen}\ (inl\ 0) \\ &= refl \end{aligned}$$

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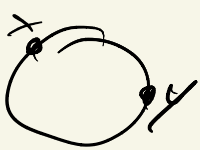
other results:

$$(x : S^1) \to (x = x) = \mathbb{Z}$$
$$(x\ y : S^1) \to \|(x = y) = \mathbb{Z}\|$$

note: 

$$((x\ y : S^1) \to x = y) \to False$$

can prove: $(x\, y : S^2) \to \| x = y \|$