Exercises on Higher Categories

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Our course should be seen as a very first introduction to the idea of higher categories. This exercise sheet is a guide for the topics that we discuss in our four lectures, and some exercises simply consist of recalling or making precise concepts and constructions discussed in the lectures.

The notion of a higher category is not one which has a single clean definition. There are numerous different approaches, of which we can only discuss a tiny fraction. In general, a very good reference is the nLab at https://ncatlab.org/nlab/show/HomePage.

Apart from that, a very incomplete list for suggested further reading is the following (all available online).

For an overview over many approaches:

- Eugenia Cheng and Aaron Lauda, *Higher-Dimensional Categories: an illustrated guide book*
  available online, `cheng.staff.shef.ac.uk/guidebook`

Introductions to simplicial sets:

- Greg Friedman, *An Elementary Illustrated Introduction to Simplicial Sets*
  very gentle introduction, `arXiv:0809.4221`.
- Emily Riehl, *A Leisurely Introduction to Simplicial Sets*
  a slightly more advanced introduction,
  `www.math.jhu.edu/~eriehl/ssets.pdf`

Quasicategories:

- Jacob Lurie, *Higher Topos Theory*
  available as paperback (ISBN-10: 0691140499) and online, `arXiv:math/0608040`

Operads (not discussed in our lectures):

- Tom Leinster, *Higher Operads, Higher Categories*
  available as paperback (ISBN-10: 0521532159) and online,

arXiv:math/0305049

Note: Exercises are ordered by topic and can be done in any order, unless an exercise explicitly refers to a previous one.

We recommend to do the exercises marked with an arrow ⇒ first. These exercises recall definitions or constructions from the lecture, and are useful to begin with in order to get some intuition. They are generally quite easy.

Apart from this, the difficulty of the exercises varies, and while the first few are easy, they are in general not ordered by difficulty. Those marked with an asterisk (or multiple asterisks) are particularly challenging or require some extra background knowledge; feel free to skip them. The exercises from 49 onwards are on material that is not covered in the lecture.

Strict 2-categories

⇒1. Recall that the category Rel has sets as objects, and a morphism $\mathsf{Rel}(A, B)$ is a *relation* between A and B, i.e. a subset of the cartesian product $A \times B$. Show that Rel can be given the structure of a strict 2-category, where 2-morphisms are *inclusions* of relations.

⇒2. For sets A, B, a *span* between A and B is a set R, together with functions $s : R \to A$ and $t : R \to B$. Show that a span between A and B can also be represented as a function $A \times B \to \mathsf{Set}$.

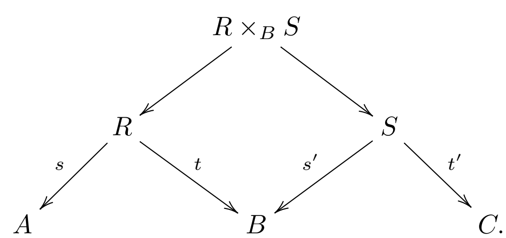
⇒3. Show that every relation determines a span where the induced map $\langle s, t\rangle : R \to A \times B$ is injective. How does the corresponding function $A \times B \to \mathsf{Set}$ from exercise 2 look like?

⇒4. Composition of spans is defined as follows. Given a span (R, s, t) between A and B, and a second span (S, s', t') between B and C, we define

$$R \times_B S \coloneqq \{(r, s) \mid t(r) = s'(s)\}.$$

($R \times_B S$ is the *pullback* of t and s' in the category of sets.)

The composition of the two spans is then given as in



Observe that composition of spans, as defined, is not strictly associative. Define the identity span, and check whether the left and right identity laws hold strictly.

⇒5. Show that a monoid is a category with one object (i.e. show that the category of monoids is equivalent to the category of (small) categories with one object).

⇒6. Let $\mathcal{C}$ be a strict 2-category, x an object of $\mathcal{C}$, and $\mathsf{id} : \mathcal{C}(x, x)$ the identity morphism on x. Let A be the set of endomorphisms of id. Prove that A is a commutative monoid.

Hints:

- first observe that all the elements of A can be composed vertically and horizontally
- use the interchange law of the 2-category $\mathcal{C}$ to show a certain distributivity property of the two compositions
- use the identity 2-cell and the distributivity property to show that $ab = b \circ a$, where ab denotes the horizontal composition of a and b, and $\circ$ denotes vertical composition
- conclude that the two operations are equal and commutative

This proof is often called the *Eckmann-Hilton argument*.

7. Show that the forgetful functor from categories to sets has a left adjoint Δ and a right adjoint ∇. Show that Δ has a left adjoint. Do the same for the forgetful functor from strict 2-categories to categories.

8. A *strict functor* between strict 2-categories $\mathcal{C}$ and $\mathcal{D}$ is given by functions that map all the data of $\mathcal{C}$ to the data of $\mathcal{D}$, preserving all the structure. Make this definition precise, and show that strict 2-categories, together with strict functors, form a category.

9. Give a definition of *strict natural transformation* between strict functors, and use it to show that strict 2-categories form a strict 2-category.

10. *Let $\mathcal{V}$ be a category with finite products. A $\mathcal{V}$-category (or *category enriched over* $\mathcal{V}$) is a generalisation of the notion of category where the homsets are replaced by objects of $\mathcal{V}$. Make this definition precise, and show that strict 2-categories are Cat-categories. Show that a cartesian closed category can be regarded as a category enriched over itself.

11. Show that the category of strict 2-categories has finite products.

12. *Define a strict $(n+1)$-category as a category enriched over strict n-categories, define the notion of strict $(n+1)$-functor and show that the category of strict $(n+1)$-categories has finite products. **Show that strict n-categories form a strict $(n+1)$-category.

13. Recall that if $\mathcal{C}$ is a category, and a an object of $\mathcal{C}$, the *slice category* C/a has morphisms $x \to a$ as objects, and commutative triangles as morphisms.

Define the corresponding notion for strict 2-categories.

Bicategories

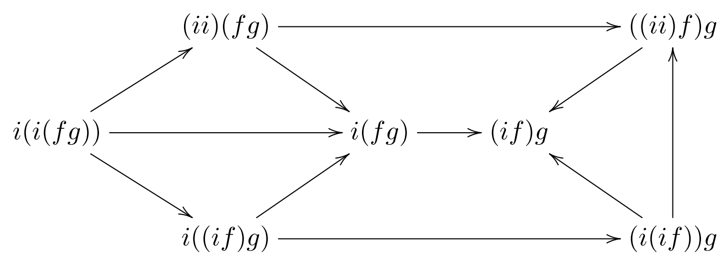
⇒14. In exercise 4, we have seen that composition of spans is not strictly associative. Thus, spans will not form a strict 2-category. Define a bicategory of spans.

15. Let $\mathcal{C}$ be a category with pullbacks. Define a bicategory of *spans* in $\mathcal{C}$.

16. *Show that in any bicategory the two unitor isomorphisms $\mathsf{id} \circ \mathsf{id} \to \mathsf{id}$ are equal.

    Hints:

    - consider the following diagram of associators and unitors (where we omit the composition operator and we write i for id):

      show that the bottom left square commutes, and deduce that the isomorphism $\mathsf{id} \circ (f \circ g) \to (\mathsf{id} \circ f) \circ g \to f \circ g$ is equal to λ

    - use naturality of the unitors and the previous result to write the composition $\lambda^{-1} \circ \rho$ as a composition of identities

⇒17. For a bicategory $\mathcal{C}$, and an object a of $\mathcal{C}$, define the slice bicategory $\mathcal{C}/a$ (*hint*: you will need to make use of the coherence properties of $\mathcal{C}$).

18. Adapt the Eckmann-Hilton argument to a general bicategory. Is the conclusion the same?

19. Recall that a *monad* on a category $\mathcal{A}$ is an endofunctor of $\mathcal{A}$, equipped with natural transformations μ, ν, satisfying certain laws. Generalise this notion to a bicategory $\mathcal{C}$ and an object a of $\mathcal{C}$. A monad in the conventional sense is then a monad *in* Cat. (*hint*: a monad is an endomorphism of a, together with two 2-cells $\mu, \nu \dots$).

    Note that we say a monad *in* a bicategory, but *on* an object (or category, in the case where the bicategory is Cat).

20. Similarly to the previous exercise, generalise the notion of an adjunction from Cat to a general bicategory (*hint*: use the definition in terms of unit and counit).

21. What is a monad *in* the bicategory of spans? How about spans of a general category with pullbacks?

⇒22. Define the notion of weak functor (called *pseudofunctor*) between bicategories.

    Hints:

    - start with the definition of strict functor
    - weaken the laws involving non-strict structure by turning them into (natural) isomorphisms
    - add coherence for those

23. Show that bicategories and pseudofunctors form a category.

24. If we replace the isomorphisms of the definition of pseudofunctor with arbitrary (not necessarily invertible) 2-cells, we get the notions of *lax* and *oplax* functors, according to the direction of those cells: the choice with 2-cells $\mathsf{id} \to F(\mathsf{id})$ and $Ff \circ Fg \to F(f \circ g)$ is called lax, and the other one oplax. Work out the complete definitions.

25. If $\mathcal{C}$ is a bicategory, show that a monad in $\mathcal{C}$ is the same thing as a lax functor $1 \to \mathcal{C}$, where 1 is the trivial (i.e. terminal) strict 2-category.

26. A *monoidal category* is a bicategory with one object. Give a more direct definition of a monoidal category that does not use 2-categorical language. Show that if $\mathcal{C}$ has finite products, it can be given the structure of a monoidal category.

27. *Let k be a field. Define a monoidal structure on the category of k-vector spaces where the operation (composition of 1-cells) is given by tensor product. Show that the functor $A \otimes -$ has a right adjoint. Do the same for abelian groups (or, more generally, modules over a commutative ring).

28. *Extend the definition of $\mathcal{V}$-category to a monoidal category $\mathcal{V}$.

29. *Show that a $\mathcal{V}$-category is the same thing as a lax functor from a codiscrete category to $\mathcal{V}$.

30. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Define their tensor product $\mathcal{C} \otimes \mathcal{D}$ as the category whose objects are the same as the objects of $\mathcal{C} \times \mathcal{D}$, but morphisms are given by alternating sequences of morphisms in $\mathcal{C}$ and $\mathcal{D}$. Make this definition precise, and show that it can be extended to a monoidal structure on Cat.

31. *Show that the functor $\mathcal{C} \otimes -$ of the previous exercise has a right adjoint.

32. *A *sesquicategory* is a category enriched over Cat, where Cat is considered as a monoidal category with the tensor product above. Give an elementary

definition of a sesquicategory. What is the relation between strict 2-categories and sesquicategories?

33. Define the notion of weak natural transformation (called *pseudonatural transformation*) between pseudofunctors. Do bicategories, pseudofunctors and pseudonatural transformations form a bicategory?

34. *Define the notion of *modification* between pseudonatural transformations. Use it to define what a biequivalence of bicategories is (*hint*: a modification is given by a 2-cell connecting two natural transformations for every object of the domain category, together with a coherence condition...)

35. *A *setoid* is a groupoid that is also a preorder (i.e. any two morphisms with the same source and target are equal). Show that, assuming the axiom of choice, the sub-bicategory of Cat determined by setoids is biequivalent to the category of sets (regarded as a 2-category).

Simplicial sets

⇒36. Recall the definition of the categories Δ and Δ_+: the objects of Δ and Δ_+ are natural numbers, written $[0]$, $[1]$, etc. We regard $[n]$ as the subset of natural numbers k with $0 \leq k \leq n$, with the standard order. The morphisms $\Delta([n],[m])$ are then simply defined to be monotone increasing maps $[n] \to [m]$, while $\Delta_+([n],[m])$ is the subset of $\Delta([n],[m])$ consisting of the injective maps (i.e. strictly increasing).

Show that every map $f : \Delta([n],[m])$ can be factored uniquely as a surjective monotone map followed by an injective one.

⇒37. Define the map $d_i : \Delta_+([n],[n+1])$ as the unique monotone map whose image is $[n+1] \backslash \{i\}$. Dually, define $s_i : \Delta([n+1],[n])$ as the only surjective map which satisfies $s_i(i) = s_i(i+1) = i$. We will refer to the maps d_i as *cofaces* and the s_i as *codegeneracies*.

Show that every map $f : \Delta([n],[m])$ can be written as a composition of cofaces and codegeneracies. *Prove that every such f admits a *normal form* as a composition of cofaces and codegeneracies.

⇒38. Show that coface and codegeneracies satisfy the *simplicial identities*:

$$
\begin{aligned}
d_j \circ d_i &= d_i \circ d_{j-1} && \text{for } i < j \\
s_j \circ s_i &= s_i \circ s_{j+1} && \text{for } i \leq j \\
s_j \circ d_i &= d_i \circ s_{j-1} && \text{for } i < j \\
s_j \circ d_i &= \mathsf{id} && \text{for } i = j \text{ or } i = j+1 \\
s_j \circ d_i &= d_{i-1} \circ s_j && \text{for } i > j+1
\end{aligned}
$$

⇒39. Another standard definition of a simplicial set is as follows (see Def. 3.2 in Friedman's *elementary illustrated introduction to simplicial sets*). A simplicial set consists of a sequence of sets $Y_0, Y_1, Y_2, \ldots$, and for every $n \geq 0$, functions $d_i : Y_n \to Y_{n-1}$ and $s_i : Y_n \to Y_{n+1}$, with $0 \leq i \leq n$, such that the "opposite" of the equations from the previous exercise are satisfied, i.e.

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i && \text{for } i < j \\ s_i \circ s_j &= s_{j+1} \circ s_i && \text{for } i \leq j \\ d_i \circ s_j &= s_{j-1} \circ d_i && \text{for } i < j \\ d_i \circ s_j &= \mathsf{id} && \text{for } i = j \text{ or } i = j+1 \\ d_i \circ s_j &= s_j \circ d_{i-1} && \text{for } i > j+1 \end{aligned}$$

Show that this this definition is equivalent to our definition of a simplicial set as a functor $\Delta^{\mathsf{op}} \to \mathsf{Set}$. *Hint:* You should do the previous exercises before attempting this one.

⇒40. Let X be a simplicial set (i.e. a functor $\Delta^{\mathsf{op}} \to \mathsf{Set}$). We will call the elements of X_n n-simplices. For any map $\theta : \Delta([n],[m])$, the action of X on θ will be denoted as $\theta^* : X_m \to X_n$.

Define a simplicial set Δ^n whose k-simplices are the maps $\Delta([k],[n])$. Show that a map of simplicial sets (i.e. a natural transformation) $\Delta^n \to X$ is the same thing as an n-simplex of X (*hint*: Yoneda lemma...).

41. We say that a *simplex* $x : X_n$ is *non-degenerate* if for all maps $\theta : \Delta([n],[m])$ such that x lies in the image of θ^*, θ is injective. Show that for any n-simplex x there exists a unique triple (m, y, ϕ), where m is a natural number, y is a non-degenerate m-simplex, and $\phi : \Delta([n],[m])$ is a surjective map such that $x = \phi^*(y)$.

42. For any natural number n, define the n-skeleton of a simplicial set X as the subfunctor of X generated by all the k-simplices of X for $k \leq n$. Show that this defines a comonad $\mathsf{sk}_n : \mathsf{sSet} \to \mathsf{sSet}$. *Show that sk_n has a right adjoint.

Kan complexes and quasicategories

Recall from exercise 40 that Δ^n is the simplicial set whose k-simplices are the maps $\Delta([k],[n])$. Define Λ^n_i to be the largest sub-simplicial set of Δ^n which does not contain the identity function $\mathsf{id} \in \Delta([n],[n])$, and which does not contain the unique injective function in $\Delta([n-1],[n])$ which does not have $i \in [n]$ in its image. Intuitively, this means that Λ^n_i is "Δ^n with the interior and one face removed". Λ^n_i is the abstract (n,i)-*horn*.

Assume that $X : \Delta^{\mathsf{op}} \to \mathsf{Set}$ is a simplicial set. An (n,i)-horn in X is a natural transformation $\Lambda^n_i \to X$, just as an n-simplex in X is a natural transformation

$\Delta^n \to X$ (see exercise 40). Recall that X is called a *Kan complex* or *weak ∞-groupoid* if every horn in X has a filler (i.e. every natural transformation $\Lambda^n_i \to X$ can be extended to a natural transformation $\Delta^n \to X$). Recall that X is called a *quasicategory* if every *inner* horn (where $0 < i < n$) has a filler.

⇒43. Try to draw a graphical representation of Δ^n and Λ^n_i for some n and i (you may want to choose $n \leq 3$).

⇒44. Given a category $\mathcal{C}$, the nerve $N(\mathcal{C})$ is a simplicial set where elements of the set $N(\mathcal{C})_k$ are chains of k morphisms of $\mathcal{C}$. Make the definition of the functor $N(\mathcal{C})$ precise. *Hint*: You can use exercise 39.

⇒45. Show that $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid, i.e. if and only if every morphism has an inverse. For the construction of the nerve $N(\mathcal{C})$, see exercise 44.

46. We have seen in the previous exercises that $N(\mathcal{C})$ has fillers for (some) horns. Show that these fillers are in fact always unique. Show that, for any quasicategory X where all required fillers exist uniquely, one can find a category $\mathcal{C}$ such that X is naturally isomorphic to $N(\mathcal{C})$.

⇒47. Define the abstract *boundary* $\partial\Delta^n$ as the simplicial set where k-cells are maps $\Delta([k], [n])$, excluding the identity on $[n]$ (that is, we have $\Lambda^n_i \subsetneq \partial\Delta^n \subsetneq \Delta^n$). An n-boundary in a simplicial set X is then just a natural transformation $\partial\Delta^n \to X$. For which n do in general all n-boundaries in X have fillers, if you know that X is ...

- any simplicial set?
- a Kan complex or quasicategory?
- $N(\mathcal{C})$, for some category $\mathcal{C}$?

What is the relationship between being able to fill boundaries, and being able to fill horns?

48. Let $\mathcal{C}$ be a category, and a an object of $\mathcal{C}$. Express the nerve of the slice category $\mathcal{C}/a$ in terms of the nerve of $\mathcal{C}$. Derive a definition of the slice of a quasicategory.

Extras

The following exercises can be used to acquire familiarity with the ideas behind some of the existing definitions of higher categories.

Double categories and multisimplicial sets

49. Let $\mathcal{C}$ be a category with pullbacks. An *internal category* in $\mathcal{C}$ is given by:

- an object O of $\mathcal{C}$ called the *object of objects*;
- an object A of $\mathcal{C}$ called the *object of morphisms*;

together with an "identity" and an associative "composition" operation. Make this definition precise and show that an internal category is Set is simply a small category. *(This notion will look familiar if you have solved enough of the previous exercises).*

50. An internal category in Cat is called a *double category.* Find an elementary definition of double category.

51. Show that a double category where the object of objects is a discrete category is the same as a strict 2-category.

52. Show that an internal category in simplicial sets is the same thing as a simplicial object in Cat.

53. Use the previous exercises to show that a double category is the same thing as a bisimplicial set satisfying unique horn filling conditions in both dimensions. Obtain an alternative definition of strict 2-category based on bisimplicial sets.

Strict ∞-categories

54. The *globe category* has natural numbers as objects, and exactly 2 morphisms $s, t : n \to m$ for all $m < n$, with $s \circ t = s \circ s = s$ and $t \circ s = t \circ t = t$. Verify that those conditions do indeed determine a category.

55. A *globular set* is a presheaf over the globe category. If X is a globular set, the elements of X_n are called n-cells. Show that a bicategory determines a globular set where s and t are both isomorphisms at high enough dimensions. Find an alternative definition of *strict* 2-category based on a globular set.

56. *Generalise the second part of the previous exercise to get a definition of *strict ∞-category.*

    Hint: you will need the following ingredients:

    - n different compositions of n-cells for every n;
    - units for all dimensions;
    - associativity of all compositions;
    - interchange laws.

57. Let T be a monad on a category $\mathcal{C}$ with pullbacks. We say that T is *cartesian* if it preserves pullbacks, and the naturality squares of η and μ are pullbacks. Show that the *list* monad on Set (i.e. the free monoid monad) is cartesian. Show that the monad of *multisets* (i.e. the free commutative monoid monad) is *not* cartesian.

58. Show that the forgetful functor from strict ∞-categories to globular sets has a left adjoint (giving the *free* strict ∞-category on a globular set). (*hint*: find a general expression for the formal composition of cells of a globular set).

59. Show that the monad determined by the adjunction of the previous exercise is cartesian.

Multicategories and operads

60. Let L be the *list* monad on Set. Define a bicategory of L-spans, whose objects are sets, and morphisms from A to B are given by spans of the form:

$$\begin{array}{ccc} & R & \\ \swarrow & & \searrow \\ LA & & B. \end{array}$$

Complete the definition of this bicategory.

61. Generalise the above definition to get a bicategory of T-spans for any cartesian monad T on a category $\mathcal{C}$ with pullbacks.

62. A T-multicategory is a monad in the bicategory of T-spans. An L-multicategory is often called a *plain multicategory*. Give an elementary definition of plain multicategory.

63. Suppose $\mathcal{C}$ has a terminal object 1. A T-operad is a T-multicategory structure on the object 1 of the bicategory of T-spans. Give an elementary definition of *plain operad*.

64. Let P be a T-operad. For an object X of $\mathcal{C}$, define $T^P X$ via the pullback:

$$\begin{array}{ccccc} & & T^P X & & \\ & \swarrow & & \searrow & \\ TX & & & & P \\ & \searrow & & \swarrow & \\ & & T1. & & \end{array}$$

Show that T^P defines a monad on the slice category $\mathcal{C}$. Give an elementary definition of this monad in the case where P is a plain operad.

65. An *algebra of an operad* P is defined to be an algebra of the monad T^P. Show that there exists a plain operad whose algebras are monoids.